MATH423 - Introduction to String Theory
Set Work: Sheet 9

1a. Compute the commutation relations

$$[L_m, \alpha_n^\nu] \ .$$

The basic commutations relations and the definition of the L_m are given in the lecture notes.

b.

Evaluate

$$[L_m, x_0^\mu] \ .$$

c.

Show that in the case $m + n \neq 0$

$$[L_m, L_n] = (m - n)L_{m+n} \ .$$

Note that you can use here the result from part **a**.

d.

Consider the state

$$L_{-2}|0>= \frac{1}{2}\sum_p \alpha_{-2-p} \cdot \alpha_p|0>$$

and evaluate its norm

$$||L_{-2}|0>||$$

e. In the case with $m + n = 0$

$$[L_m, L_n] = (m - n)L_{m+n} + A(m)\delta_{m+n} \ ,$$

$A(m)$ is the central charge. Consider the case with $m = 2; m+n = 0$. Evaluate the expectation value

$$<0|[L_2, L_{-2}]|0>$$

and the value of $A(2)$ using the results from **d**.

e. Similarly, Show that $A(0) = A(1) = 0$.

f. The Jacobi identity for operators A, B, and C is

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$$

Consider the Jacobi identity with $A, B, C = L_1, L_{-1-n}, L_n$. Using the commutation relations for the Virasoro operators show that

$$(2n + 1)A(1) - (n + 2)A(n) - (1 - n)A(n + 1) = 0.$$

Using $A(1) = 0$ derive a recursion relation for the central charge $A(n)$.

Show that the solution with

$$A(n) = \frac{1}{12} cn(n + 1)(n - 1)$$

Satisfies the recursion relation and use the result from **e** to fix c.

2. Due to normal–ordering ambiguities, the central extension of the Virasoro algebra has the form

$$[L_m, L_n] = (m - n)L_{m+n} + A(m)\delta_{m+n,0},$$

(a) Show that if $A(1) \neq 0$ it is possible to change the definition of L_0 by adding a constant, so that $A(1) = 0$.

(b) For $A(1) = 0$ show that the generators L_0 and $L_{\pm 1}$ form a closed algebra.

(c) Use the Jacobi identity to derive a recursion relation for the coefficients $A(m)$. You may assume that $A(2)$ is given.