MATH423 - Introduction to String Theory
Set Work: Sheet 5

1. Consider an action for a particle moving in three space dimensions with dynamical variables $q_i(t)$ and velocities $\dot{q}_i(t)$:

$$S = \int dt L(q_i(t), \dot{q}_i(t); t). \tag{1}$$

Calculate the variation δS of the action under the variation $\delta q_i(t)$ of the coordinates and derive the Euler–Lagrange equations of motion.

2. Consider the action

$$S[x,e] = \frac{1}{2} \int e d\tau \left(\frac{1}{e^2} \left(\frac{dx^\mu}{d\tau} \right)^2 - m^2 \right) , \tag{2}$$

where τ is an arbitrary parameter and $ed\tau$ is an invariant line element.

(a) Show that the action is invariant under reparameterisation of the world–line and under Poincare transformations of Minkowski space.

(b) Perform variations of x^μ and of e to obtain the equations of motions for x^μ and e respectively.

(c) For $m^2 > 0$, eliminate e by its equation of motion, and substitute the result back into (2). Show that you obtain the action for a free massive particle.

(d) Instead of eliminating e by its equation of motion, you can set it to a constant value by reparameterisation. Derive the equation of motion and the contraint equation for the two cases $m^2 > 0$ and $m^2 = 0$.

3. The action for the relativistic string is given by (with $c = 1$)

$$S_{\rm NG}[X] = \int d^2\sigma \mathcal{L} = -T \int_\Sigma d^2\sigma \sqrt{|det(\partial_\alpha X^\mu \partial_\beta X_\mu)|} \ . \tag{3}$$

(a) Show that the action is invariant under reparametrisations of the world–sheet Σ:

$$\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma^0, \sigma^1), \quad \text{where} \ \det \left(\frac{\partial \tilde{\sigma}^\alpha}{\partial \sigma^\beta} \right) > 0 \ . \tag{4}$$

(b) Compute the momentum densities

$$P^0_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} \ , P^1_\mu = \frac{\partial \mathcal{L}}{\partial X'^\mu} \ . \tag{5}$$

(c) Show that the canonical momenta $\Pi^\mu = P_0^\mu$ are subject to the two constraints

$$\begin{aligned} \Pi_\mu X'_\mu &= 0 \\ \Pi^2 + T^2 (X')^2 &= 0 \end{aligned}$$

and that the Hamiltonian vanishes

$$\mathcal{H} = \dot{X}\Pi - \mathcal{L} = 0 \tag{6}$$