MATH423 String Theory Solutions 9

solution to problem 1.

1a.

The basic commutator between oscillators is

$$[\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0}$$

Compute:

$$\begin{aligned}
[L_m, \alpha_n^\nu] &= \frac{1}{2}\sum_{k=-\infty}^{\infty}\eta_{\rho\sigma}[\alpha^\rho_{m-k}\alpha^\sigma_k, \alpha^\nu_n] \\
&= \frac{1}{2}\sum_{k=-\infty}^{\infty}\eta_{\rho\sigma}(\alpha^\rho_{m-k}[\alpha^\sigma_k, \alpha^\nu_n] + [\alpha^\rho_{m-k}, \alpha^\nu_n]\alpha^\sigma_k) \\
&= \frac{1}{2}\sum_{k=-\infty}^{\infty}\eta_{\rho\sigma}(\alpha^\rho_{m-k}k\eta^{\sigma\nu}\delta_{k+n,0} + (m-k)\eta^{\rho\nu}\delta_{m-k+n,0}\alpha^\sigma_k) \\
&= \frac{1}{2}(-n\alpha^\nu_{m+n} + (-n)\alpha^\nu_{m+n}) \\
&= -n\alpha^\nu_{m+n}
\end{aligned}$$

(I assumed that $m \neq 0$, so that there is no need to worry about normal ordering.)

1b.

For the closed string

$$L_m = \frac{1}{2}\sum_n \alpha_n^\mu \alpha_{(m-n)_\mu}$$

with

$$\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu = \frac{1}{2}l_s p^\mu$$

Using

$$[AB, C] = A[B, C] + [A, C]B$$

and

$$[\alpha_n^\mu, x_0^\nu] = -i\sqrt{\frac{\alpha'}{2}}\eta^{\mu\nu}\delta_{n,0} \tag{1}$$

we get

$$
\begin{aligned}
[L_m, x_0^\nu] &= [\frac{1}{2}\sum_n \alpha_n^\mu \alpha_{(m-n)_\mu}, x_0^\mu] \\
&= \frac{1}{2}\sum_n \left(\alpha_n^\mu [\alpha_{(m-n)_\mu}, x_0^\nu] + [\alpha_n^\mu, x_0^\nu]\alpha_{(m-n)_\mu}\right) \\
&= \frac{1}{2}\sum_n \left(-i\sqrt{\frac{\alpha'}{2}}\eta_\mu^\nu \delta_{m-n,0}\right) + \left(-i\sqrt{\frac{\alpha'}{2}}\eta^{\mu\nu}\delta_{n,0}\alpha_{(m-n)_\mu}\right) \\
&= -i\alpha_m^\mu \sqrt{\frac{\alpha'}{2}}
\end{aligned}
$$

and similarly for $\tilde{L}_m$.

1c.

Using

$$[A, BC] = [A, B]C + B[A, C]$$

we have

$$
\begin{aligned}
[L_m, L_n] &= \frac{1}{2}\sum_p [L_m, \alpha^\mu_{(n-p)}\alpha_{p_\mu}] \\
&= \frac{1}{2}\sum_p ([L_m, \alpha^\mu_{(n-p)}]\alpha_{p_\mu} + \alpha^\mu_{(n-p)}[L_m, \alpha_{p_\mu}] \\
&= \frac{1}{2}\sum_p -(n-p)\alpha^\mu_{(m+n-p)}\alpha_{p_\mu} + \alpha^\mu_{(n-p)}(-p)\alpha_{(m+p)_\mu}] \\
&= \frac{1}{2}\sum_p \left((-n+p)\alpha^\mu_{m+n-p}\alpha_{p_\mu} - p\alpha^\mu_{n-p}\alpha_{(m+p)_\mu}\right) \\
&\quad \text{letting p} \to \text{p} - \text{m in the second sum we get} \\
&= \frac{1}{2}\sum_p \left((-n+p)\alpha^\mu_{m+n-p}\alpha_{p_\mu} + (-p+m)\alpha^\mu_{m+n-p}\alpha_{(p)_\mu}\right) \\
&= (m-n)L_{m+n}
\end{aligned}
$$

1d.

The only term in the infinite sum that contributes to the state is the one with $p = 1$:

$$L_{-2}|0> = \frac{1}{2}\alpha_{-1}\cdot\alpha_{-1}|0>$$

imposing the commutation relations on the α operators we get

$$||L_{-2}|0>|| =< 0|\frac{1}{2}\alpha_1^\mu \alpha_{1,\mu}\alpha_{-1}^\nu \alpha_{-1,nu}|0>= \frac{1}{2}\eta_{\mu\nu} eta^{\mu\nu} = \frac{1}{2}D \neq 0. \quad (2)$$

This clearly contradicts our expectation that

$$L_n|\text{phys}>= 0$$

i.e. that the Fourier modes of the world–sheet energy–momentum tensor annihilates the vacuum. The problem arises because of the nontrivial commutation relations $[L_m, L_n]$ when $m = -n$.

1e.

From **d**, and using $< 0|L_0|0 >= 0$, we have

$$< 0|L_2 L_{-2}|0 >= \frac{1}{2}D \quad , \quad < 0|L_{-2}L_2|0 >= 0$$
$$\Longrightarrow \quad < 0|[L_2, L_{-2}]|0 >= \frac{1}{2}D$$

Hence $A(2) = \frac{1}{2}D$

For $A(0)$ this is obvious as we have

$$A(0) = [L_0, L_0] = 0.$$

For

$$L_{-1} = \frac{1}{2}\sum_p \alpha_{-1+p}^\mu \alpha_{-p\mu}.$$

The only possible nonvanishing term in

$$L_{-1}|0>$$

comes from

$$\alpha_0 \alpha_{-1}|0> .$$

However, we are taking the state with $p^\mu|0 >= 0$ and hence $< 0|[L_1 L_{-1}]|0 >= 0$ and $A(1) = 0$.

1f.

From the Jacobi identity we have:

$$[L_k, [L_n, L_m]] + [L_n, [L_m, L_k]] + [L_m, [L_k, L_n]] = 0 \quad (3)$$

Inserting the commutation relation

$$[L_m, L_n] = (m-n)L_{m+n} + A(m)\delta_{m+n} \tag{4}$$

into (3) we get

$$(n-m)A(k) + (m-k)A(n) + k-n)A(m) = 0 \tag{5}$$

Taking $k = 1$, $m = -1-n$,$n = n$ we have,

$$(2n+1)A(1) - (n+2)A(n) + (1-n)A(-1-n) = 0 \tag{6}$$

From (4) it follows that $A(-m) = -A(m)$ therefore

$$(2n+1)A(1) - (n+2)A(n) - (1-n)A(n+1) = 0 \tag{7}$$

Using $A(1) = 0$ it follows that

$$A(n+1) = \frac{(n+2)}{(n-1)}A(n) \tag{8}$$

Inserting the proposition

$$A(n) = \frac{1}{12}cn(n+1)(n-1) \tag{9}$$

into (8) we see that the recursion relation is indeed satisfied.

Taking $n = 2$ in (9) and using $A(2) = \frac{1}{2}D$ we find that

$$c = D.$$

The algebra (4) is the celebrated Virasoro algebra and the central charge $A(m)$ embodies the quantum anomaly.

solution to problem 2.

a Assume that $A(1) \neq 0$. We begin by noting that $A(1)$ only appears in a single commutation relation.

$$[L_1, L_{-1}] = 2L_0 + A(1). \tag{10}$$

Modifying L_0 by

$$\tilde{L}_0 = L_0 + \frac{1}{2}A(1) \tag{11}$$

gives

$$[L_1, L_{-1}] = 2\tilde{L}_0. \tag{12}$$

Since $A(1)$ is central, this redefinition does not affect the commutator structure. We can therefore always choose L_0 such that $A(1) = 0$.

b We choose L_0 such that $A(1) = 0$. The commutation relations for L_{-1}, L_0 and L_1 are

$$\begin{aligned} [L_1, L_{-1}] &= 2L_0, \\ [L_1, L_0] &= L_1, \\ [L_0, L_{-1}] &= L_{-1}. \end{aligned}$$

Therefore, $\{L_{-1}, L_0, L_1\}$ form a closed algebra.

c Evaluating the Jacobi identity, we find that

$$\begin{aligned} 0 &= [[L_m, L_n], L_p] + [[L_p, L_m], L_n] + [[L_n, L_p], L_m] \\ &= (m-n)[L_{m+n}, L_p] + (p-m)[L_{p+m}, L_n] + (n-p)[L_{n+p}, L_m] \\ &= \{(m-n)(m+n-p) + (p-m)((p+m-n) + (n-p)(n+p-m)\} L_{m+n+p} \\ &\quad + \{(m-n)A(m+n) + (p-m)A(p+m) + (n-p)A(n+p)\} \delta_{m+n+p,0}. \end{aligned}$$

Since the generators $\{L_k\}$ and the central elements are linearly independent, both coefficients must vanish separately. A little algebra shows that the coefficient of L_{m+n+p} is indeed zero. Requiring that the coefficient of the central element vanishes, we obtain an equation for $A(k)$

$$(m-n)A(m+n) - (2m+n)A(-n) + (m+2n)A(-m) = 0. \tag{13}$$

We first note that the skew symmetry of the commutator implies that $A(-k) = -A(k)$.

$$\begin{aligned} [L_n, L_m] &= -[L_m, L_n] \\ (n-m)L_{n+m} + A(n)\delta_{m+n,0} &= -(m-n)L_{m+n} - A(m)\delta_{m+n,0} \\ A(-m) &= -A(m) \end{aligned}$$

Assume $A(1) = 0$. For $n = 1$ we obtain a recursion relation for $A(k)$ from (13).

$$(m-1)A(m+1) = (m+2)A(m) \tag{14}$$

For a given $A(2)$, this recurrence relation uniquely determines $A(k)$.