MATH423 String Theory Solutions 5

1.

Under a variation $\delta q_i(t)$ of the coordinate, the variation of the velocity is $\frac{d\delta q_i}{dt}$. Since the Lagrangian L depends on q_i and $\dot{q}_i$ the full variation is

$$\delta S = \int dt \left\{ \frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\frac{d\delta q_i}{dt} \right\}. \tag{1}$$

The second term in the brackets is rewritten in terms of a total derivative and a term proportional to δq_i

$$\delta S = \int dt \left\{ \frac{\partial L}{\partial q_i}\delta q_i + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i \right\}. \tag{2}$$

The total time derivative vanishes when we set the variations to vanish at the initial and final times. The variation δS is therefore

$$\delta S = \int dt \delta q_i(t) \left\{ \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \right\}. \tag{3}$$

If this variation is to vanish for all $\delta q_i(t)$ we must have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0. \tag{4}$$

These are the Euler–Lagrange equations of motion for the coordinates q_i.

2.

The action in terms of $\tilde{\tau}(\tau)$ is given by

$$S[\tilde{x},\tilde{e}] = \frac{1}{2}\int \tilde{e} d\tilde{\tau} \left(\frac{1}{\tilde{e}^2}\left(\frac{d\tilde{x}^\mu}{d\tilde{\tau}}\right)^2 - m^2 \right) \ , \tag{5}$$

where $\tilde{\tau}(\tau)$ is an arbitrary parameter and $\tilde{e}d\tilde{\tau}$ is an invariant line element. Hence, the following hold:

$$\begin{aligned}
\tilde{e}d\tilde{\tau} &= ed\tau \\
ed\tau &= \tilde{e}\frac{d\tilde{\tau}}{d\tau}d\tau \\
\tilde{e}(\tilde{\tau}) &= e(\tau)\frac{d\tau}{d\tilde{\tau}} \\
\frac{d}{d\tau} &= \left(\frac{d\tilde{\tau}}{d\tau}\right)\frac{d}{d\tilde{\tau}} \\
\tilde{x}^\mu(\tilde{\tau}) &= x^\mu(\tau)
\end{aligned}$$

Hence the integrand of the action transforms as

$$\begin{aligned}
&\tilde{e}(\tilde{\tau})d\tilde{\tau}\left(\frac{1}{\tilde{e}(\tilde{\tau})^2}\left(\frac{d\tilde{x}^\mu}{d\tilde{\tau}}\right)^2 - m^2\right) \\
=\;& e(\tau)\frac{d\tau}{d\tilde{\tau}}\frac{d\tilde{\tau}}{d\tau}d\tau\left(\frac{1}{(e(\tau))^2\left(\frac{d\tau}{d\tilde{\tau}}\right)^2}\left(\frac{dx^\mu}{d\tau}\right)^2\left(\frac{d\tau}{d\tilde{\tau}}\right)^2 - m^2\right) \\
=\;& e(\tau)d\tau\left(\frac{1}{e(\tau)^2}\left(\frac{dx^\mu}{d\tau}\right)^2 - m^2\right)
\end{aligned}$$

and is invariant under reparametization $\tau \to \tilde{\tau}$. Invariance under Poincare transformations follows from the fact that it is constructed from four vectors and the fact $\dot{x}^\mu \dot{x}_\mu$ and m^2 are Lorentz and Poincare invariants. The fact that the action contains only derivatives of x^μ also implies that it is invariant under constant four vector shifts.

b.

We obtain the equations of motion from the Euler–Lagrange equations with

$$L = \frac{1}{2}\left(\frac{1}{e}\left(\frac{dx^\mu}{d\tau}\right)^2 - m^2 e\right)$$

Hence,

$$\begin{aligned}
e \;:\; \frac{d}{d\tau}\frac{\partial L}{\partial\left(\frac{\partial e}{\partial \tau}\right)} - \frac{\partial L}{\partial e} &= -\frac{\partial L}{\partial e} = -\frac{1}{2}\left(-\frac{1}{e^2}\left(\frac{dx^\mu}{d\tau}\right)^2 - m^2\right) = 0 \\
&\Longrightarrow \left(\frac{dx^\mu}{d\tau}\right)^2 + e^2 m^2 = 0 \\
x^\mu \;:\; \frac{d}{d\tau}\frac{\partial L}{\partial\left(\frac{\partial x^\mu}{\partial \tau}\right)} - \frac{\partial L}{\partial x^\mu} &= \frac{d}{d\tau}\left(\frac{1}{e}\frac{dx^\mu}{d\tau}\right) = 0
\end{aligned}$$

c. For $m^2 > 0$ $(dx^\mu/d\tau)^2 < 0$

$$\Rightarrow e^2 > 0 \quad \Rightarrow \quad e = \frac{\sqrt{-\left(\frac{dx^\mu}{d\tau}\right)^2}}{m}$$

Substituting back into the action we get

$$
\begin{aligned}
S &= \frac{1}{2}\int d\tau \left(\frac{(-m)(-\dot{x}^2)}{\sqrt{-\dot{x}^2}} - \sqrt{-\dot{x}^2} m \right) \\
&= -m \int d\tau \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}
\end{aligned}
$$

which is the action for a free massive particle.

d. The e equation of motion $\dot{x}^2 + e^2 m^2 = 0$

- For $m^2 > 0$ impose $e = \dfrac{1}{m}$

The equation of motions become

$$
\frac{d}{d\tau}\left(\frac{dx^\mu}{d\tau}\right) = \frac{d^2 x^\mu}{d\tau^2} = 0
$$

$$
\dot{x}^2 = \eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -1
$$

- For $m^2 = 0$ impose $e = 1$

The equation of motions become

$$
\frac{d}{d\tau}\frac{dx^\mu}{d\tau} = \frac{d^2 x^\mu}{d\tau^2} = 0
$$

$$
\dot{x}^2 = 0
$$

3.

a

Under reparameterization the measure of integration transforms as

$$d\sigma^0 d\sigma^1 = |\det\left(\frac{\partial\sigma^i}{\partial\tilde{\sigma}^j}\right)| d\tilde{\sigma}^0 d\tilde{\sigma}^1 = |\det M| d\tilde{\sigma}^0 \tilde{\sigma}^1$$

where $M = [M_{ij}]$ is the matrix defined by $M_{ij} = (\partial\sigma^i/\partial\tilde{\sigma}^j)$. Similarly,

$$d\tilde{\sigma}^0 d\tilde{\sigma}^1 = |\det\left(\frac{\partial\tilde{\sigma}^i}{\partial\sigma^j}\right)| d\sigma^0 d\sigma^1 = |\det\tilde{M}| d\sigma^0 d\sigma^1$$

where $\tilde{M} = [\tilde{M}_{ij}]$ is the matrix defined by $\tilde{M}_{ij} = (\partial\tilde{\sigma}^i/\partial\sigma^j)$. We note that

$$|\det M||\det\tilde{M}| = 1.$$

From invariance of the infinitesimal line element under reparameterization

$$g_{ij}(\sigma)d\sigma^i d\sigma^j = \tilde{g}_{ij}(\tilde{\sigma})d\tilde{\sigma}^i d\tilde{\sigma}^j = \tilde{g}_{pq}(\tilde{\sigma})\frac{\partial\tilde{\sigma}^p}{\partial\sigma^i}\frac{\partial\tilde{\sigma}^q}{\partial\sigma^j}d\sigma^i d\sigma^j$$

we deduce that the

$$g_{ij}(\sigma) = \tilde{g}_{pq}(\tilde{\sigma})\frac{\partial\tilde{\sigma}^p}{\partial\sigma^i}\frac{\partial\tilde{\sigma}^q}{\partial\sigma^j} \ .$$

or in matrix form

$$g_{ij}(\sigma) = \tilde{g}_{pq}(\tilde{\sigma})\tilde{M}_{pi}\tilde{M}_{qj} = \left(\tilde{M}^T_{ip}\right)\tilde{g}_{pq}\tilde{M}_{qj} \ . \tag{6}$$

We consider a target space surface $\mathcal{S}$ described by the mappting $X^\mu(\sigma^1,\sigma^2)$. Given a vector dX^μ tangent to the surface, left $|ds|$ denote its length. Then we can write

$$|ds^2| = |dX^\mu dX_\mu| = \left|\frac{\partial X^\mu}{\partial\sigma^\alpha}d\sigma^\alpha\frac{\partial X_\mu}{\partial\sigma^\beta}d\sigma^\beta\right| = \left|\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X_\mu}{\partial\sigma^\beta}d\sigma^\alpha d\sigma^\beta\right|$$

The induced matrix is given by

$$g_{\alpha\beta} = \frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X_\mu}{\partial\sigma^\beta},$$

whose determinant appears in the square root of the Nambu–Gotto action. Taking the determinant of both sides of (6), and defining,

$$g \equiv \det(g_{ij})$$

we have

$$g = (\det\tilde{M}^T)\tilde{g}(\det\tilde{M}) = \tilde{g}(\det\tilde{M})^2.$$

or

$$\sqrt{g} = \sqrt{\tilde{g}}|\det\tilde{M}|$$

Combing the results we get

$$\int d^2\sigma\sqrt{g} = \int d^2\tilde{\sigma}|\det M|\sqrt{\tilde{g}}|det\tilde{M}| = \int d^2\tilde{\sigma}\sqrt{\tilde{g}}$$

b

Identifying $\sigma^0 \equiv \tau$, the world–sheet 'time' parameter and $\sigma^1 \equiv \sigma$, the world–sheet internal parameter, we obtain from the Nambu–Gotto action:

$$S = \int_{\tau_i}^{\tau_f} d\tau L = \int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\sigma \mathcal{L}(\dot{X}^\mu, X^{\mu\prime}) \ ,$$

where $\mathcal{L}$ is given by

$$\mathcal{L}(\dot{X}^\mu, X^{\mu\prime}) = -T\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2} \ .$$

Hence,

$$\Pi_\mu = P^0_\mu = \frac{\partial\mathcal{L}}{\partial\dot{X}^\mu} \ , = -T\frac{(\dot{X}\cdot X')X'_\mu - (X')^2\dot{X}_\mu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}}$$

and

$$P^1_\mu = \frac{\partial\mathcal{L}}{\partial X'^\mu} \ . = -T\frac{(\dot{X}\cdot X')\dot{X}_\mu - (\dot{X})^2X'_\mu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}}$$

c

$$\Pi^\mu X'_\mu = T\frac{(X')^2(\dot{X}\cdot X') - (\dot{X}\cdot X')(X')^2}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}} = 0$$

$$\begin{aligned}\Pi^\mu\Pi_\mu &= T^2\frac{\left[(\dot{X}\cdot X')X'_\mu - (X')^2\dot{X}_\mu\right]^2}{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2} \\ &= T^2\frac{(\dot{X})^2(X')^4 - 2(\dot{X}\cdot X')^2(X')^2 + (\dot{X}\cdot X')^2(X')^2}{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2} = -T^2(X')^2\end{aligned}$$

$$\implies \quad \Pi^\mu \Pi_\mu + T^2 (X')^2 = 0.$$

$$\begin{aligned}
\mathcal{H} &= \dot{X}^\mu \Pi_\mu - \mathcal{L} \\
&= \dot{X}^\mu \frac{\left(T\left((X')^2 \dot{X}_\mu - (\dot{X}\cdot X')X'_\mu\right)\right)}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2 (X')^2}} + T\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2 (X')^2} \\
&= \frac{T\left((X')^2 \dot{X}^2 - (X'\cdot \dot{X})^2 + (X'\cdot \dot{X})^2 - (X')^2 \dot{X}^2\right)}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2 (X')^2}} = 0
\end{aligned}$$