MATH423 String Theory Solutions 10

1a.

The nontrivial Lorentz transformations for this boost are:

$$\begin{aligned} x'^0 &= \gamma(x^0 - \beta x^1) \\ x'^1 &= \gamma(x^1 - \beta x^0) \end{aligned}$$

We then find

$$\begin{aligned} x'^{\pm} = \frac{x'^0 \pm x'1}{\sqrt{2}} &= \frac{\gamma}{\sqrt{2}}(x^0 - \beta x^1 \pm x^1 \mp \beta x^0) \\ &= \frac{\gamma(1 \mp \beta)}{\sqrt{2}}(x^0 \pm x^1) = \gamma(1 \mp \beta) x^{\pm} \end{aligned}$$

Therefore

$$x'^+ = \sqrt{\frac{1-\beta}{1+\beta}} x^+ \quad , \quad x'^- = \sqrt{\frac{1+\beta}{1-\beta}} x^- \quad , \quad x'^2 = x^2 \quad , \quad x'^3 = x^3 \qquad (1)$$

The light cone coordinates x^+ and x^- do not mix under boosts along X^1!

b.

The new rotated coordinates are given by

$$\begin{aligned} x'^0 &= x^0 \\ x'^1 &= \cos\theta x^1 + \sin\theta x^2 \\ x'^2 &= -\sin\theta x^1 + \cos\theta x^2 \\ x'^3 &= x^3 \end{aligned}$$

After some algrebra we find

$$\begin{aligned} x'^+ &= \frac{1}{2}(1+\cos\theta)x^+ + \frac{1}{2}(1-\cos\theta)x^- + \frac{\sin\theta}{\sqrt{2}}x^2 \\ x'^- &= \frac{1}{2}(1-\cos\theta)x^+ + \frac{1}{2}(1+\cos\theta)x^- - \frac{\sin\theta}{\sqrt{2}}x^2 \\ x'^2 &= -\frac{\sin\theta}{\sqrt{2}}x^+ + \frac{\sin\theta}{\sqrt{2}}x^- + \cos\theta x^2 \end{aligned}$$

c. For a boost with velocity parameter β along x^3, the Lorentz transformations are:

$$\begin{aligned} x'^0 &= \gamma(x^0 - \beta x^3) \\ x'^1 &= x^1 \\ x'^2 &= x^2 \\ x'^3 &= \gamma(x^3 - \beta x^0) \end{aligned}$$

Therefore,

$$\begin{aligned} x'^+ = \frac{x'^0 + x'^1}{\sqrt{2}} &= \frac{\gamma(x^0 - \beta x^3) + x^1}{\sqrt{2}} \\ &= \frac{\gamma}{2}(x^+ + x^-) - \frac{\gamma\beta x^3}{\sqrt{2}} + \frac{1}{2}(x^+ - x^-). \end{aligned}$$

Similalry for x'^-, x'^2 and x'^3 we get

$$\begin{aligned} x'^+ &= \frac{1}{2}(\gamma+1)x^+ + \frac{1}{2}(\gamma-1)x^- - \frac{\gamma\beta}{\sqrt{2}}x^3, \\ x'^- &= \frac{1}{2}(\gamma-1)x^+ + \frac{1}{2}(\gamma+1)x^- - \frac{\gamma\beta}{\sqrt{2}}x^3, \\ x'^2 &= x^2 \\ x'^3 &= \gamma x^3 - \frac{\gamma\beta}{\sqrt{2}}(x^+ + x^-). \end{aligned}$$

2a.

$$\begin{pmatrix} x \\ ct \end{pmatrix} \sim \begin{pmatrix} x \\ ct \end{pmatrix} + 2\pi \begin{pmatrix} R \\ -R \end{pmatrix}. \tag{2}$$

We use (2) to compute

$$\begin{aligned} x^+ &= \frac{ct + x}{\sqrt{2}} \sim \frac{1}{\sqrt{2}}(ct - 2\pi R + x + 2\pi R) = x^+ \\ x^- &= \frac{ct - x}{\sqrt{2}} \sim \frac{1}{\sqrt{2}}(ct - 2\pi R - x - 2\pi R) = x^- - 2\pi(\sqrt{2}R) \end{aligned}$$

We therefore have the light –cone identifications

$$x^+ \sim x^+ \ , \ x^- x^- - 2\pi(\sqrt{2}R).$$

The compactification does not involve light–cone time.

b. With the Lorentz transformation $x' = \gamma(x - \beta ct)$ and $ct' = \gamma(ct - \beta x)$, we have

$$\begin{aligned} x' \sim \gamma(x + 2\pi R - \beta[ct - 2\pi R]) &= x' + 2\pi R\gamma(1+\beta) \\ ct' \sim \gamma(ct - 2\pi R - \beta[x + 2\pi R]) &= ct' - 2\pi R\gamma(1+\beta) \end{aligned}$$

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} \sim \begin{pmatrix} x' \\ ct' \end{pmatrix} + 2\pi\sqrt{\frac{1+\beta}{1-\beta}} \begin{pmatrix} R \\ -R \end{pmatrix}.$$

Comparing with (2) we see that in the boosted frame the effective radious of compactification is increased.

c.

Considering the compactification

$$\begin{pmatrix} x \\ ct \end{pmatrix} \sim \begin{pmatrix} x \\ ct \end{pmatrix} + 2\pi \begin{pmatrix} \sqrt{R^2 + R_s^2} \\ -R \end{pmatrix}. \tag{3}$$

We search for a fame (x', ct')

$$\begin{aligned} x' &= \gamma(x - \beta ct) \\ ct' &= \gamma(ct - \beta x) \end{aligned}$$

in which this compactification is standard. For this we need $ct' \sim ct'$. So,

$$\begin{aligned} ct' &\sim \gamma(ct - 2\pi R - \beta[x + 2\pi\sqrt{R^2 + R_s^2}]) \\ &= ct' - \gamma(2\pi)(R + \beta\sqrt{R^2 + R_s^2}) \end{aligned}$$

Thus, we need $R + \beta\sqrt{R^2 + R_s^2} = 0$, so

$$\beta = -\frac{R}{\sqrt{R^2 + R_s^2}}.$$

Now examine the x' identification to find the radius:

$$\begin{aligned} x' &\sim \gamma(x + 2\pi\sqrt{R^2+R_s^2} - \beta(ct - 2\pi R)) \\ x' &\sim x' + 2\pi\gamma(\sqrt{R^2+R_s^2} + \beta R) \end{aligned}$$

Similarly,

$$\begin{aligned} \gamma(\sqrt{R^2+R_s^2} + \beta R) &= \frac{1}{\sqrt{1-\beta^2}}(\sqrt{R^2+R_s^2} - \frac{R^2}{\sqrt{R^2+R_s^2}}) \\ &= \frac{\sqrt{R^2+R_s^2}}{R_s}\left(\frac{R^2+R_s^2-R^2}{\sqrt{R^2+R_s^2}}\right) = R_s \end{aligned}$$

Thus, $x' \sim x' + 2\pi R_s$. The radiius of compactification is R_s. It is interesting to note that $\gamma = \sqrt{R^2+R_s^2}/R_s$. In the limit as R_s is small this is a very large Lorentz factor $\gamma \sim R/R_s$.

3.

In the formula

$$\ell_{\text{vac}} = \rho_{\text{vac}}^{\alpha}\hbar^{\beta}c^{\gamma} \tag{4}$$

The units on both sides must agree:

$$L = \left(\frac{M}{L^3}\right)^{\alpha}\left(\frac{ML^2}{T}\right)^{\beta}\left(\frac{L}{T}\right)^{\gamma}$$

This gives three equations for matching powers of M, L and T:

$$\alpha + \beta = 0, \quad -3\alpha + 2\beta + \gamma = 1, \quad -\beta - \gamma = 0 \quad .$$

The solution is $\alpha = -1/4$, $\beta = 1/4$ and $\gamma = -1/4$. This means that (4) gives

$$\ell_{\text{vac}} = \left(\frac{\hbar}{c\rho_{\text{vac}}}\right)^{\frac{1}{4}}$$

Taking $\rho_{\text{vac}} = 7.7 \times 10^{-27} kg/m^3$ gives

$$\ell_{\text{vac}} = 8.22 \times 10^{-5} m = 82.2\mu\text{m}$$