MATH423 - Introduction to String Theory
Set Work: Sheet 8

1. The Fourier modes of the energy–momentum tensor

For a closed string, the Fourier expansion of the solution to the equations of motion is

$$X^\mu(\sigma) = x^\mu + \frac{1}{\pi T} p^\mu \sigma^0 + \frac{i}{2}\sqrt{\frac{1}{\pi T}} \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-2in\sigma^-} + \sum_{n\neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-2in\sigma^+}$$

The lightcone components of the energy–momentum tensor are

$$T_{\pm\pm} = \frac{1}{2}(\partial_\pm X^\mu)^2.$$

Use the Fourier expansion of X^μ to show that the Fourier modes of T_{--} at worldsheet time $\sigma^0 = 0$ are given by:

$$L_m = T \int_0^\pi d\sigma^1 e^{-2im\sigma^1} T_{--} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n \ .$$

2. Compute the commutation relations

$$[L_m, \alpha_n^\nu] \ .$$

The basic commutations relations and the definition of the L_m are given in the lecture notes.

3. Show that the operators

$$l_m = \frac{i}{2} e^{2im\sigma^+} \partial_+$$

satisfy the Witt algebra

$$[l_m, l_n] = (m-n) l_{m+n} \ .$$