MATH423 - Introduction to String Theory
Set Work: Sheet 6

1. The Polyakov action is given by:

$$S_{\mathrm{P}} = -\frac{T}{2}\int d^2\sigma\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu}\ , \tag{1}$$

where $h = \det(h_{\alpha\beta})$.

(a) Show that the Polyakov action is invariant under reparameterisations $\sigma^\alpha \to \sigma^\alpha(\sigma^0, \sigma^1)$. Use that reparametrisations act by

$$\tilde{X}^\mu(\tilde{\sigma}) = X^\mu(\sigma) \quad \text{and} \quad \tilde{h}_{\alpha\beta}(\tilde{\sigma}) = \frac{\partial\sigma^\gamma}{\partial\tilde{\sigma}^\alpha}\frac{\partial\sigma^\delta}{\partial\tilde{\sigma}^\beta}h_{\gamma\delta}(\sigma) \tag{2}$$

on the fields.

(b) Show that the Polyakov action is invariant under Weyl transformations

$$h_{\alpha\beta}(\sigma) \to \mathrm{e}^{\Lambda(\sigma)}h_{\alpha\beta}(\sigma)\ . \tag{3}$$

Why does this not work if you replace the string by a particle or membrane?

(c) In the conformal gauge $h_{\alpha\beta} = \eta_{\alpha\beta}$, the energy–momentum tensor takes the form

$$T_{\alpha\beta} = \partial_\alpha X^\mu\partial_\beta X_\mu - \frac{1}{2}\eta_{\alpha\beta}\eta^{\gamma\delta}\partial_\gamma X^\mu\partial_\delta X_\mu\ . \tag{4}$$

Show that

$$\begin{aligned} \eta^{\alpha\beta}T_{\alpha\beta} &= 0 \\ \partial^\alpha T_{\alpha\beta} &= 0 \end{aligned}$$

2. For a 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad \delta A = \begin{pmatrix} \delta a_{11} & \delta a_{12} \\ \delta a_{21} & \delta a_{22} \end{pmatrix} \tag{5}$$

show that

$$\delta\mathrm{detA} = \mathrm{detATr}(A^{-1}\delta A)\ .$$

Hence, derive the world–sheet energy momentum tensor $T_{\alpha\beta}$ from the Polyakov action, which is defined by

$$T_{\alpha\beta} = -\frac{2}{T}\frac{1}{\sqrt{-h}}\frac{\delta S}{\delta h^{\alpha\beta}}$$