MATH423 - Introduction to String Theory
Set Work: Sheet 5

1. Consider the action

$$S[x,e] = \frac{1}{2}\int e d\tau \left(\frac{1}{e^2}\left(\frac{dx^\mu}{d\tau}\right)^2 - m^2 \right) , \tag{1}$$

where τ is an arbitrary parameter and $ed\tau$ is an invariant line element.

(a) Show that the action is invariant under reparameterisation of the world–line and under Poincare transformations of Minkowski space.

(b) Perform variations of x^μ and of e to obtain the equations of motions for x^μ and e respectively.

(c) For $m^2 > 0$, eliminate e by its equation of motion, and substitute the result back into (1). Show that you obtain the action for a free massive particle.

(d) Instead of eliminating e by its equation of motion, you can set it to a constant value by reparameterisation. Derive the equation of motion and the contraint equation for the two cases $m^2 > 0$ and $m^2 = 0$.

2. The action for the relativistic string is given by (with $c = 1$)

$$S_{\rm NG}[X] = \int d^2\sigma \mathcal{L} = -T\int_\Sigma d^2\sigma \sqrt{|det(\partial_\alpha X^\mu \partial_\beta X_\mu)|} \; . \tag{2}$$

(a) Show that the action is invariant under reparametrisations of the world–sheet Σ:

$$\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma^0, \sigma^1), \quad \text{where } \det\left(\frac{\partial\tilde{\sigma}^\alpha}{\partial\sigma^\beta}\right) > 0 \; . \tag{3}$$

(b) Compute the momentum densities

$$P^0_\mu = \frac{\partial\mathcal{L}}{\partial \dot{X}^\mu} \; , P^1_\mu = \frac{\partial\mathcal{L}}{\partial X'^\mu} \; . \tag{4}$$

(c) Show that the canonical momenta $\Pi^\mu = P^\mu_0$ are subject to the two constraints

$$\begin{aligned} \Pi_\mu X'_\mu &= 0 \\ \Pi^2 + T^2(X')^2 &= 0 \end{aligned}$$

and that the Hamiltonian vanishes

$$\mathcal{H} = \dot{X}\Pi - \mathcal{L} = 0 \tag{5}$$