

MATH423 - Introduction to String Theory
Set Work: Sheet 4

1. If the path of a point particle is parameterised by proper time, the equation of motion of a free particle is

$$\frac{d^2x}{ds^2} = 0. \quad (1)$$

Consider a new parameter $\tau = f(s)$. Find the most general function f for which (1) implies

$$\frac{d^2x}{d\tau^2} = 0. \quad (2)$$

2. Consider the point particle action given by

$$S = -mc \int_{\tau_i}^{\tau_f} \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} d\tau .$$

Vary the action to find a manifestly reparameterization invariant form of the free particle equation of motion.

3. The relativistic version of Newton's second law is

$$\frac{d\vec{p}}{dt} = \frac{d}{dt} \left(\frac{m\vec{v}}{\sqrt{1 - \vec{v}^2}} \right) = \vec{F}.$$

Show that this can be written in manifestly covariant form

$$\frac{dp^\mu}{ds} = f^\mu$$

and find the relation between \vec{F} and the relativistic force vector f^μ .

4. The action for a non-relativistic particle of mass m and charge q coupled to an electromagnetic field is obtained by replacing the first term in

$$S = -mc \int_{\mathcal{P}} ds + \frac{q}{c} \int_{\mathcal{P}} A_\mu(x) dx^\mu .$$

by the non-relativistic action for a free point particle:

$$S = \int \frac{1}{2} m v^2 dt + \frac{q}{c} \int A_\mu(x) \frac{dx^\mu}{dt} dt .$$

Time t is used to parameterize the second integral.

(a) Rewrite the action S in terms of the potentials (Φ, \vec{A}) and the ordinary velocity \vec{v} . What is the Lagrangian?

(b) Calculate the canonical momentum \vec{p} conjugate to the position of the particle and show that it is given by

$$\vec{p} = m\vec{v} + \frac{q}{c}\vec{A}.$$

(c) Construct the Hamiltonian for the charged particle and show that it is given by

$$H = \frac{1}{2m} \left(\vec{p} - \frac{q}{c}\vec{A} \right)^2 + q\Phi.$$

5. The action for a relativistic particle of mass m and charge q coupled to an electromagnetic field is given by

$$S = -mc \int_{\mathcal{P}} ds + \frac{q}{c} \int_{\mathcal{P}} d\tau A_{\mu}(x(\tau)) \frac{dx^{\mu}}{d\tau}(\tau).$$

Derive the equation of motion from this action by taking the variation $x^{\mu}(\tau) \rightarrow x^{\mu}(\tau) + \delta x^{\mu}(\tau)$.