MATH423 String Theory Solutions 8

1.

For a closed string, the Fourier expansion of the solution to the equations of motion is

$$\begin{aligned}
X^\mu(\sigma) &= x^\mu + \frac{1}{\pi T} p^\mu \sigma^0 + \frac{i}{2}\sqrt{\frac{1}{\pi T}} \left(\sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-2in\sigma^-} + \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-2in\sigma^+} \right) \\
&= x^\mu + \frac{1}{2\pi T} p^\mu \sigma^+ + \frac{1}{2\pi T} p^\mu \sigma^- + \frac{i}{2}\sqrt{\frac{1}{\pi T}} \left(\sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-2in\sigma^-} + \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-2in\sigma^+} \right)
\end{aligned}$$

Defining $\alpha_0 = \tilde{\alpha}_0 = p^\mu/(2\sqrt{\pi T})$, from the mode expansion ot $X^\mu(\sigma)$ we have

$$\begin{aligned}
\partial_- X^\mu &= \frac{i}{2}\frac{1}{\sqrt{\pi T}} \sum_n \frac{1}{n} \alpha_n (-2in \mathrm{e}^{-i2n\sigma^-}) \\
&= \frac{1}{\sqrt{\pi T}} \sum_n \alpha_n \mathrm{e}^{-i2n\sigma^-}
\end{aligned}$$

$$\begin{aligned}
(\partial_- X)^2 &= \frac{1}{\pi T} \sum_{n,m} \alpha_n^\mu \alpha_{m\mu} \mathrm{e}^{-i2(n+m)\sigma^-} \\
&= \frac{1}{\pi T} \sum_{n,p} \alpha_n \cdot \alpha_{p-n} \mathrm{e}^{-i2p\sigma^-}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad & \text{At } \sigma^0 = 0 \\
& T \int_0^\pi d\sigma^1 \mathrm{e}^{-2im\sigma^1} \frac{1}{2} (\partial_- X)^2 = \\
& \frac{T}{\pi T} \frac{1}{2} \sum_{n,p} \alpha_n \alpha_{p-n} \int_0^\pi d\sigma^1 \mathrm{e}^{-i2(m-p)\sigma^1} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_n \cdot \alpha_{m-n}
\end{aligned}$$

2.

The basic commutator between oscillators is

$$[\alpha_m^\mu, \alpha_n^\nu] = m \eta^{\mu\nu} \delta_{m+n,0}$$

Compute:

$$
\begin{aligned}
[L_m, \alpha_n^\nu] &= \frac{1}{2}\sum_{k=-\infty}^{\infty} \eta_{\rho\sigma}[\alpha_{m-k}^\rho \alpha_k^\sigma, \alpha_n^\nu] \\
&= \frac{1}{2}\sum_{k=-\infty}^{\infty} \eta_{\rho\sigma}(\alpha_{m-k}^\rho [\alpha_k^\sigma, \alpha_n^\nu] + [\alpha_{m-k}^\rho, \alpha_n^\nu]\alpha_k^\sigma) \\
&= \frac{1}{2}\sum_{k=-\infty}^{\infty} \eta_{\rho\sigma}(\alpha_{m-k}^\rho k\eta^{\sigma\nu}\delta_{k+n,0} + (m-k)\eta^{\rho\nu}\delta_{m-k+n,0}\alpha_k^\sigma) \\
&= \frac{1}{2}(-n\alpha_{m+n}^\nu + (-n)\alpha_{m+n}^\nu) \\
&= -n\alpha_{m+n}^\nu
\end{aligned}
$$

(I assumed that $m \neq 0$, so that there is no need to worry about normal ordering.)

3.

Evaluate commutator:

$$
\begin{aligned}
[l_m, l_n] &= \left[\frac{i}{2}\mathrm{e}^{2im\sigma^+}\partial_+, \frac{i}{2}\mathrm{e}^{2in\sigma^+}\partial_+\right] \\
&= \frac{i^2}{4}\mathrm{e}^{2im\sigma^+}\partial_+\left(\mathrm{e}^{2in\sigma^+}\right)\partial_+ - \frac{i^2}{4}\mathrm{e}^{2in\sigma^+}\partial_+\left(\mathrm{e}^{2im\sigma^+}\right)\partial_+ \\
&+ \frac{i^2}{4}\mathrm{e}^{2i(m+n)\sigma^+}\partial_+^2 - \frac{i^2}{4}\mathrm{e}^{2i(m+n)\sigma^+}\partial_+^2 \ .
\end{aligned}
$$

The terms in the last line cancel. The terms in the line above can be simplified to give the desired result:

$$
\begin{aligned}
[l_m, l_n] &= \frac{i^2}{4}2i(n-m)\mathrm{e}^{2i(m+n)\sigma^+}\partial_+ \\
&= \frac{i}{2}(m-n)\mathrm{e}^{2i(m+n)\sigma^+}\partial_+ \\
&= (m-n)l_{m+n} \ .
\end{aligned}
$$