MATH423 String Theory Solutions 7

1.

(a).
Constraints: $\dot{X} \cdot X' = 0$ and $\dot{X}^2 - X'^2 = 0$
For the given solution:

$$\begin{aligned} \dot{X}^0 &= L \quad \dot{X}^1 = -L\cos\sigma^1 \sin\sigma^0 \quad \dot{X}^2 = +L\cos\sigma^1\cos\sigma^0 \\ X'^0 &= 0 \quad X'^1 = -L\sin\sigma^1\cos\sigma^0 \quad X'^2 = -L\sin\sigma^1\sin\sigma^0 \end{aligned}$$

plugging into the constraints

$$\dot{X}^\mu X'_\mu = L^2\cos\sigma^1\sin\sigma^1\sin\sigma^0\cos\sigma^0 - L^2\cos\sigma^1\sin\sigma^1\sin\sigma^0\cos\sigma^0 = 0$$

$$\begin{aligned} \dot{X}^2 - X'^2 &= -L^2 + L^2\cos^2\sigma^1\sin^2\sigma^0 + L^2\cos^2\sigma^1\cos^2\sigma^0 + \\ & \quad L^2\sin^2\sigma^1\cos^2\sigma^0 + L^2\sin^2\sigma^1\sin^2\sigma^0 + \\ &= -L^2 + L^2(\cos^2\sigma^1 + \sin^2\sigma^1) = 0 \end{aligned}$$

The wave equation

$$\ddot{X}^\mu - X^{\mu\prime\prime} = 0$$

follows similarly by plugging-in the given solution.

At time σ^0 the string is a straight line along the X^1 axis extending from $X^1 = L$ at $\sigma^1 = 0$ to $X^1 = -L$ at $\sigma^1 = \pi$. Hence the length is $2\pi L$. The time dependence is such that the string rotates rigidly around the point $(X^1, X^2) = (0,0)$ in the (X^1, X^2)-plane.

(b). The relativistic momentum density of a string is $\Pi^\mu = T\partial_0 X^\mu$, where T is the string tension.

$$(\Pi^\mu) = TL(1, -\sin\sigma^0\cos\sigma^1, \cos\sigma^0\cos\sigma^1, 0, \ldots, 0) \tag{3}$$

The momentum is obtained by integrating this over the string:

$$\begin{aligned} P^\mu &= \int_0^\pi d\sigma^1 \Pi^\mu \\ &= TL(\pi, 0, 0, \ldots, 0) \end{aligned} \tag{4}$$

The mass is $M^2 = -P^\mu P_\mu$:

$$M^2 = T^2 L^2 \pi^2 \tag{5}$$

The angular momentum density in (X^1, X^2)-plane is

$$\begin{aligned} j_{12} &= T(X^1 \partial_0 X^2 - X^2 \partial_0 X^1) \\ &= TL^2 \cos^2 \sigma^1 \end{aligned} \tag{6}$$

The total angular momentum in the (X^1, X^2)-plane is obtained by integration:

$$\begin{aligned} J_{12} &= \int_0^\pi d\sigma^1 j_{12} \\ &= TL^2 \frac{\pi}{2} = \alpha' M^2 \end{aligned} \tag{7}$$

where M is the mass and $\alpha' = (2\pi T)^{-1}$ is the Regge parameter.

(c). The velocity of the endpoint at $\sigma^1 = 0$ is

$$v^\mu = \frac{dX^\mu}{dt} \tag{8}$$

where $t = X^0$ is the observer time measured in the Lorentz frame that we use to describe the motion of the string. The relation between observer time $t = X^0$ and world sheet time σ^0 is given by the solution, i.e., $t = X^0 = L\sigma^0$. Hence

$$\begin{aligned} v^\mu &= \partial_0 X^\mu \frac{\partial \sigma^0}{\partial t} = L^{-1} \partial_0 X^\mu \\ &= (1, -\sin\sigma^0 \cos\sigma^1, \cos\sigma^0 \cos\sigma^1, 0, \ldots, 0) \\ &= (1, -\sin\sigma^0, \cos\sigma^0, 0, \ldots) \end{aligned} \tag{9}$$

where we used that $\sigma^1 = 0$ in the last line. Thus the relativistic velocity is light-like:

$$v^\mu v_\mu = -1 + \sin^2\sigma^0 + \cos^2\sigma^0 = -1 + 1 = 0 \tag{10}$$

showing that the endpoint moves with the speed of light (we have set $c = 1$). Alternaltively, the spatial part of the velocity is

$$\sum_{i=1}^{D-1} (v_i)^2 = 1 (= c) \tag{11}$$

For the other endpoint $\sigma = \pi$ we find the same result (inspection of the components of v^μ shows that this endpoint moves in the opposite direction. This is clear from our discussion of the solution).

To show that the endpoints move with the speed of light for any open string solution, we note that the second constraint, $\partial_0 X^\mu \partial_0 X_\mu + \partial_1 X^\mu \partial_1 X_\mu = 0$, when evaluated at an endpoint subject to the boundary condition $\partial_1 X^\mu = 0$ becomes $\partial_0 X^\mu \partial_0 X_\mu = 0$. Since $\frac{\partial t}{\partial \sigma^0} = \frac{\partial X^0}{\partial \sigma^0} > 0$ (both $t = X^0$ and σ^0 are time-like coordinates) this implies that

$$\frac{\partial X^\mu}{\partial t}\frac{\partial X_\mu}{\partial t} = 0 \tag{12}$$

which means that the relativistic velocity of an endpoint is light-like. Hence the endpoints always move with the speed of light.

2.

1. follows by plugging-in the solution
2. At $\sigma^0 = 0$ the string describes a circle of radius R in the (X^1, X^2)-plane. Between $\sigma^0 = 0$ and $\sigma^0 = \pi/4$, this circle shrinks and forms a single point, located at $X^i = 0$ at $\sigma^0 = \pi/4$. From $\sigma^0 = \pi/4$ to $\sigma^0 = \pi/2$ the string re–expands into a circle of radius R. The points of the string are located at antipodal points compared to $\sigma^0 = 0$. The circle then shrinks and re-expands again. At $\sigma^0 = \pi$ every point is at the same position as at $\sigma^0 = 0$. The motion is periodic with period π (in the time-like coordinate σ^0).
3. The length l of the string is

$$\begin{aligned} l &= \int_0^\pi d\sigma^1 \sqrt{(\partial_1 X^1)^2 + (\partial_1 X^2)^2} \\ &= R\cos(2\sigma^0) \int_0^\pi d\sigma^1 \sqrt{4\cos^2\sigma^1 + 4\sin^2\sigma^1} = 2\pi R\cos(2\sigma^0) \end{aligned} \tag{13}$$

The relativistic momentum density:

$$\Pi^\mu = T\partial_0 X^\mu = TR(2, -2\sin 2\sigma^0 \cos 2\sigma^1, -2\sin 2\sigma^0 \sin 2\sigma^1, 0, \ldots, 0) \tag{14}$$

The relativistic momentum

$$p^\mu = \int_0^\pi d\sigma^1 \Pi^\mu = 2RT(\pi, 0, \dots, 0) \tag{15}$$

Thus the energy, measured in our reference frame is $E = p^0 = 2\pi RT$. This is also the mass, i.e., the energy measured in the rest frame, because

$$M^2 = -p^\mu p_\mu = 4\pi^2 R^2 T^2 \tag{16}$$

This is clear, because our reference frame is the rest frame of the center of mass of the string.

4. At $\sigma^0 = 0$, we have $\partial_0 X^i = 0$ for $i = 1, 2, \dots, D-1$, meaning that all points of the string are at rest. Above we found that the energy is $E = 2\pi RT$ for all times σ^0. At $\sigma^0 = 0$ the string is a circle of radius R and length $l_0 = 2\pi R$. Hence, at $\sigma^0 = 0$ we $E = l_0 T$ or energy = T $\times$ length. This leads to the interpretation of T as the tension of the string, i.e, its energy per length. Note that the total energy is only given by the product of tension and length if the string is at rest. If the string moves it has kinetic energy in addition to the potential energy given by tension $\times$ length. In the above solution total energy is conserved, but during the time evolution potential energy gets converted into kinetic energy and vice versa. If the string has maximal radius it only has potential energy, if it has radius zero it only has kinetic energy.
The open string discussed in the previous example always has a non-vanishing kinetic energy, because its endpoints move with the speed of light.