MATH423 String Theory Solutions 6

1. The Polyakov action is given by:

$$S_{\rm P} = -\frac{T}{2}\int d^2\sigma\sqrt{h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu}\ , \tag{1}$$

where $h = -\det(h_{\alpha\beta}) = |\det(h_{\alpha\beta})|$.

We already saw in the case of the Nambu–Gotto action (see solution to set 5) that the measure of integration times $\sqrt{-\det(h_{\alpha\beta})}$ is invariant under reparameterization. We are given that X_μ and $h_{\alpha\beta}$ transforms as

$$\tilde{X}^\mu(\tilde{\sigma}) = X^\mu(\sigma) \quad\text{and}\quad \tilde{h}_{\alpha\beta}(\tilde{\sigma}) = \frac{\partial\sigma^\gamma}{\partial\tilde{\sigma}^\alpha}\frac{\partial\sigma^\delta}{\partial\tilde{\sigma}^\beta}h_{\gamma\delta}(\sigma) \tag{2}$$

Since $h^{\alpha\beta}h_{\beta\gamma} = \tilde{h}^{\alpha\beta}\tilde{h}_{\beta\gamma} = \delta^\alpha_\gamma$ we have

$$\tilde{h}^{\alpha\beta}(\tilde{\sigma}) = \frac{\partial\tilde{\sigma}_\gamma}{\partial\sigma_\alpha}\frac{\partial\tilde{\sigma}_\delta}{\partial\sigma_\beta}h^{\gamma\delta}(\sigma) \tag{3}$$

Then using the chain rule we have

$$\frac{\partial X^\mu}{\partial\tilde{\sigma}^\alpha}\frac{\partial X^\nu}{\partial\tilde{\sigma}^\beta}\eta_{\mu\nu} \ \rightarrow\ \frac{\partial X^\mu}{\partial\sigma^\gamma}\frac{\partial\sigma^\gamma}{\partial\tilde{\sigma}^\alpha}\frac{\partial X^\nu}{\partial\sigma^\delta}\frac{\partial\sigma^\delta}{\partial\tilde{\sigma}^\beta}\eta_{\mu\nu} \tag{4}$$

Combining (3) and (4) we indeed see that

$$h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu}$$

is invariant under reparameterization.

b. since

$$h_{\alpha\beta}(\sigma) \rightarrow {\rm e}^{\Lambda(\sigma)}h_{\alpha\beta}(\sigma)\ .$$

$$h^{\alpha\beta}(\sigma) \rightarrow {\rm e}^{-\Lambda(\sigma)}h^{\alpha\beta}(\sigma)\ . \tag{5}$$

$$\det(h_{\alpha\beta}) \rightarrow {\rm e}^{2\Lambda(\sigma)}\det(h_{\alpha\beta}) \tag{6}$$

Hence we see that under the transformations the exponents cancel and the Polyakov action is invariant under Weyl transformations.

In the case of a particle with d world–volume dimensions the tranformation of the square toot of the determinant will priduce a factor

$$\mathrm{e}^{\frac{d}{2}\Lambda(\sigma)}$$

it is obvious that this can only cancel the exponent arising from the inverse metric when $d = 2$, *i.e.* for a string. For any $d \neq 2$ the factors do not cancel and the action is not invariant under Weyl transformations.

c.

$$\begin{aligned} \eta^{\alpha\beta}T_{\alpha\beta} &= \eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}\eta^{\alpha\beta}\eta_{\alpha\beta}\eta^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X_\mu \\ &= \eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}\cdot 2 \cdot \eta^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X_\mu = 0 \end{aligned} \tag{7}$$

$$\begin{aligned} \partial^\alpha T_{\alpha\beta} &= \partial^\alpha(\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}\eta_{\alpha\beta}\eta^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X_\mu) \\ &= (\partial^\alpha\partial_\alpha X^\mu)\partial_\beta X_\mu + \partial_\alpha X^\mu(\partial^\alpha\partial_\beta X_\mu) \\ &\quad -\frac{1}{2}\eta_{\alpha\beta}\eta^{\gamma\delta}(\partial^\alpha\partial_\gamma X^\mu)\partial_\delta X_\mu \\ &\quad -\frac{1}{2}\eta_{\alpha\beta}\eta^{\gamma\delta}\partial_\gamma X^\mu(\partial^\alpha\partial_\delta X_\mu) \\ &= 0 + \partial_\alpha X^\mu(\partial^\alpha\partial_\beta X_\mu) \\ &\quad -\frac{1}{2}(\partial_\beta\partial^\delta X^\mu)\partial_\delta X_\mu \\ &\quad -\frac{1}{2}\partial^\delta X^\mu(\partial_\beta\partial_\delta X_\mu) \\ &= \partial_\alpha X^\mu(\partial^\alpha\partial_\beta X_\mu) \\ &\quad -\frac{1}{2}(\partial^\alpha\partial_\beta X^\mu)\partial_\alpha X_\mu \\ &\quad -\frac{1}{2}\partial_\alpha X^\mu(\partial_\beta\partial^\alpha X_\mu) = 0. \end{aligned}$$

where we have used the rule $\partial_\alpha(fg) = g(\partial_\alpha f) + f(\partial_\alpha g)$ and $\eta^{\alpha\beta}\partial_\beta = \partial^\alpha$, etc.

2.

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad \delta A = \begin{pmatrix} \delta a_{11} & \delta a_{12} \\ \delta a_{21} & \delta a_{22} \end{pmatrix} \tag{8}$$

$$\text{det}A = a_{11}a_{22} - a_{12}a_{21}$$

$$\delta(\text{det}A) = (\delta a_{11})a_{22} + a_{11}(\delta a_{22}) - (\delta a_{12})a_{21} - a_{12}(\delta a_{21})$$

$$A^{-1} = \frac{1}{\text{det}A} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \tag{9}$$

$$\begin{aligned}(\text{detA})(A^{-1}\delta A) \quad = & \\ & \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \cdot \begin{pmatrix} \delta a_{11} & \delta a_{12} \\ \delta a_{21} & \delta a_{22} \end{pmatrix} = \\ & \begin{pmatrix} a_{22}\delta a_{11} - a_{12}\delta a_{21} & a_{22}\delta a_{12} - a_{12}\delta a_{22} \\ -a_{21}\delta a_{12} + a_{11}\delta a_{21} & -a_{21}\delta a_{12} + a_{11}\delta a_{22} \end{pmatrix}\end{aligned}$$

$$\Rightarrow \ \text{Tr} = a_{22}\delta a_{11} + a_{11}\delta a_{22} - a_{21}\delta a_{12} - a_{12}\delta a_{21} = \delta(\text{det}A)$$

b. With $h = \text{det}h_{\alpha\beta}$ and using the result from **a** we have

$$\delta h = h(h^{\alpha\beta}\delta h_{\beta\alpha})$$

$$\delta\sqrt{-h} = \frac{1}{2}\sqrt{-h}(h^{\alpha\beta}\delta h_{\beta\alpha})$$

The variation of $\delta h_{\alpha\beta}$ in terms of $\delta h^{\alpha\beta}$ is obtained from the relation

$$h^{\alpha\beta}h_{\alpha\beta} = 2$$

$$\begin{aligned} &\Rightarrow \ \delta(h^{\alpha\beta})h_{\alpha\beta} + h^{\alpha\beta}\delta(h_{\alpha\beta}) = 0 \\ &\Rightarrow \ h^{\alpha\beta}\delta(h_{\alpha\beta}) = -\delta(h^{\alpha\beta})h_{\alpha\beta} \ \ \& \ \ \text{with} \ \ \delta(h_{\alpha\beta}) = \delta(h_{\beta\alpha}) \\ &\Rightarrow \ \delta h = -hh_{\alpha\beta}\delta(h^{\alpha\beta}) \\ &\Rightarrow \ \delta(\sqrt{-h}) = -\frac{1}{2}\frac{1}{\sqrt{-h}}\delta h = -\frac{1}{2}\frac{(-h)h_{\alpha\beta}\delta h^{\alpha\beta}}{\sqrt{-h}} = -\frac{1}{2}\sqrt{-h}h_{\alpha\beta}\delta h^{\alpha\beta} \end{aligned}$$

Finally, the variation of the action with respect to $h^{\alpha\beta}$ gives

$$\begin{aligned}\delta S &= -\frac{T}{2}\int d^2\sigma\sqrt{-h}\left(-\frac{1}{2}h_{\alpha\beta}\delta h^{\alpha\beta}(h^{\gamma\eta}\partial_\gamma X^\mu\partial_\eta X_\mu) + \delta h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu\right) \\ &= -\frac{T}{2}\int d^2\sigma\sqrt{-h}\delta^{\alpha\beta}\left(\partial_\alpha X^\mu\partial_\beta X_\mu - \frac{1}{2}h_{\alpha\beta}(h^{\gamma\eta}\partial_\gamma X^\mu\partial_\eta X_\mu)\right) \ .\end{aligned}$$

Hence

$$T_{\alpha\beta} = -\frac{2}{T}\frac{1}{\sqrt{-h}}\frac{\delta S}{\delta h^{\alpha\beta}} = \partial_\alpha X^\mu\partial_\beta X_\mu - \frac{1}{2}h_{\alpha\beta}(h^{\gamma\eta}\partial_\gamma X^\mu\partial_\eta X_\mu) \ .$$