MATH423 String Theory Solutions 3

1a.

$$T_{\lambda\mu\nu} = \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu}.$$

where

$$F_{\mu\nu} - \partial_\mu A_\nu - \partial_\nu A_\mu.$$

$$T_{0ij} = \partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i}.$$

For time independent fields $\partial_0 F_{ij} = 0$, so $T_{0ij} = 0$ implies

$$\partial_i F_{0j} - \partial_j F_{0i} = 0 \to \partial_i E_j - \partial_j E_i = 0 \ . \tag{1}$$

$\vec{E} = -\vec{\nabla}\Phi$ says $E_i = -\partial_i \Phi$, so condition (1) is satisfied since

$$\partial_i \partial_j \Phi = \partial_j \partial_i \Phi.$$

b.

With d spatial dimensions and a point charge q at $\vec{x} = 0$, the magnitude $E(r)$ of the electric field is

$$E(r) = \frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}} \frac{q}{r^{d-1}} \quad \text{(derived in lectures)} \tag{2}$$

Since $E(r) = -\frac{d\Phi}{dr}(r)$.

$$\Phi(r) = \frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}} \frac{1}{d-2} \frac{q}{r^{d-2}} \tag{3}$$

Setting $\Phi = 0$ at $r = \infty$ for $d > 2$. Using $\Gamma(x+1) = x\Gamma(x)$,

$$\Gamma\left(\frac{d}{2}\right) = \left(\frac{d}{2} - 1\right) \Gamma\left(\frac{d}{2} - 1\right) = \frac{1}{2}(d-2)\Gamma\left(\frac{d}{2} - 1\right) \tag{4}$$

so we get

$$\Phi(r) = \frac{\Gamma(\frac{d}{2} - 1)}{2\pi^{\frac{d}{2}}} \frac{q}{r^{d-2}} \qquad d > 2. \tag{5}$$

2.

a.

The standard Bohr radius

$$a_0 = \frac{\hbar^2}{m_e e^2} = 5.29 \times 10^{-9}\text{cm}, \quad \text{arises from the potential} \quad V = -\frac{e^2}{r}. \quad (6)$$

In the gravitational case, the potential is

$$V = -Gm_e m_p/r,$$

so the gravitational Bohr radius a_0^g is obtained by replacing e^2 with $Gm_e m_p$:

$$a_0^g = \frac{\hbar^2}{m_e(Gm_e m_p)} = \frac{\hbar^2}{m_e e^2}\left(\frac{e^2}{Gm_e m_p}\right) . \quad (7)$$

The ratio between the "gravitational" and the conventional Bohr radius is therefore

$$\frac{a_0^g}{a_0} = \frac{e^2}{Gm_e m_p} = \frac{(4.8 \times 10^{-10})^2}{(6.67 \times 10^{-8})(0.911 \times 10^{-27})(1.672 \times 10^{-24})} \approx 2.27 \times 10^{39}. \quad (8)$$

This gives

$$a_0^g = 1.2 \times 10^{31}\text{cm} \quad (9)$$

Since 1 light year $\approx\ 9.5 \times 10^{17}$cm, $a_0^g = 1.3 \times 10^{13}$ light years!! The universe is about 10^{10} years old, so this distance is quite large.

b.

We introduce factors of G, c and $\hbar$ into the formula, with exponents α, β and γ to be determined:

$$kT = \frac{1}{8\pi M} \rightarrow \frac{G^\alpha c^\beta \hbar^\gamma}{8\pi M}. \quad (10)$$

Since $[kT] = [E] = ML^2/T^2$, using $[G] = L^3M^{-1}T^{-2}$, $[c] = LT^{-1}$, and $[\hbar] = ML^2T^{-1}$, we find

$$\frac{ML^2}{T^2} = L^{3\alpha+\beta+2\gamma}M^{-\alpha+\gamma-1}T^{-2\alpha-\beta-\gamma} \quad (11)$$

This gives three equations for three unknowns

$$3\alpha + \beta + 2\gamma = 2, \quad -\alpha + \gamma - 1 = 1, \quad -2\alpha - \beta - \gamma = -2 \quad . \quad (12)$$

The solution is $\alpha = -1$, $\beta = 3$ and $\gamma = 1$. As a result

$$kT = \frac{\hbar c^3}{8\pi G M} = \left(\frac{\hbar c}{G}\right) \frac{c^2}{8\pi M} = \frac{m_P^2 c^2}{8\pi M}, \tag{13}$$

where $m_P = 2.17 \times 10^{-5}$gr is the Planck mass. For a mass with million solar masses $M = 10^6 M_{sun} \approx 2 \times 10^{39}$gr,

$$kT = \frac{(2.17 \times 10^{-5})^2 (3 \times 10^{10})^2}{8\pi 2 \times 10^{39}} = 8.43 \times 10^{-30}\text{eV} \quad \Rightarrow \quad T \approx 6.1 \times 10^{-14}\text{K}. \tag{14}$$

For a room temperature black hole, $T \approx 300$K, and we find $M = \frac{6.1\times 10^{-14}\text{K}}{300\text{K}} 2 \times 10^{39}$gr $\approx 4.1 \times 10^{20}$kg, which is approximately $6.8 \times 10^{-5} M_{\text{earth}}$.

3.

$$\frac{d^2y_1}{dx^2} + \frac{\mu_1}{T_0}w^2 y_1(x) = 0 \quad \text{for} \quad 0 \le x \le a,$$
$$\frac{d^2y_2}{dx^2} + \frac{\mu_2}{T_0}w^2 y_2(x) = 0 \quad \text{for} \quad a \le x \le 2a.$$

Here $y_1(x) = y(x)$ for $x \in [0, a]$ and $y_2(x) = y(x)$ for $x \in [a, 2a]$. We let

$$\lambda_1^2 = \frac{\mu_1}{T_0}w^2 \quad \text{and} \quad \lambda_2^2 = \frac{\mu_2}{T_0}w^2. \tag{15}$$

with this notation, the two equations take similar form

$$\frac{d^2y_i}{dx^2} + \lambda_i^2 y_i(x) = 0, \quad \lambda_i = \sqrt{\frac{\mu_i}{T_0}}w, \ i = 1, 2. \tag{16}$$

Since $y(x = 0) = y(x = 2a) = 0$ we write

$$y_1(x) = A\sin(\lambda_1 x), \quad \text{and} \quad y_2(x) = B\sin(\lambda_2(2a - x)), \tag{17}$$

which satisfy the differential equation and the boundary conditions at $x = 0$ and $x = 2a$.

a.

Since the string does not break at $x = a$, we must have $y_1(a) = y_2(a)$. Since there is no mass at the joining point the slopes must agree too: $y_1'(a) = y_2'(a)$.

b.

Applying the boundary conditions in (a) we find

$$\begin{aligned} A\sin\lambda_1 a &= B\sin\lambda_2 a \\ A\lambda_1\cos\lambda_1 a &= -B\lambda_2\cos\lambda_2 a \end{aligned}$$

These two equations can be written as

$$\begin{pmatrix} \sin\lambda_1 a & -\sin\lambda_2 a \\ \lambda_1\cos\lambda_1 a & \lambda_2\cos\lambda_2 a \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0\ . \tag{18}$$

Since we need solutions where both A and B are not zero, the determinant of the above matrix must vanish:

$$\lambda_2 \sin \lambda_1 a \cos \lambda_2 a + \lambda_1 \cos \lambda_1 a \sin \lambda_2 a = 0 \ . \tag{19}$$

This equation can be used to find the oscillations frequencies. As a check assume $\mu_1 = \mu_2 = \mu_0$, in which case $\lambda_1 = \lambda_2 \equiv \lambda$. The condition becomes $\sin(2\lambda a) = 0$, which gives $\lambda(2a) = n\pi$. This gives the familiar frequencies

c.

With $\mu_1 = \mu_0$ and $\mu_2 = 2\mu_0$ we write

$$\lambda_1 = \sqrt{\frac{\mu_0}{T_0}} w \equiv \lambda \quad \text{and} \quad \lambda_2 = \sqrt{\frac{2\mu_0}{T_0}} w = \sqrt{2}\lambda \ .$$

Eq (19) then gives

$$\sqrt{2} \sin \lambda a \cos \sqrt{2}\lambda a + \cos \lambda a \sin \sqrt{2}\lambda a = 0 \ .$$

Solving this equation numerically we find that to the lowest root is $\lambda a \approx 1.26567$. This gives the lowest frequency

$$w_L = \sqrt{\frac{T_0}{\mu_0}} \frac{1.26567}{a} = \sqrt{\frac{T_0}{\mu_0}} \frac{2.53134}{2a}.$$

Note that w_L is bounded as

$$\sqrt{\frac{T_0}{\mu_0}} \frac{\pi/\sqrt{2}}{(2a)} < w_L < \sqrt{\frac{T_0}{\mu_0}} \frac{\pi}{(2a)} \ .$$

The lower and upper bounds are the lowest frequencies of strings of contant mass density $2\mu_0$ and μ_0 respectively ($\pi/\sqrt{2} \approx 2.22144$).