**MATH423 String Theory Solutions 1**

1.

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Table 1: The weight lattice of the spinorial 16 representation of $SO(10)$. Each entry should be multiplied by the $\frac{1}{2}$. The $\pm\frac{1}{2}$ entries are the charges with respect to the five $U(1)$ generators of the Cartan subalgebra. The product of the five charges should be either positive or negative. If we choose to be positive then we can have either, zero, two or four negative charges (or spins) of the total five. Alternatively, if we choose the product to be negative we can have either one, three or five of the charges (or spins) to have a minus sign. In either case the total number of possibilities is 16. These are the two spinorial representations of $SO(10)$ being the chiral 16 and the anti–chiral $\overline{16}$. The table above shows the chiral 16 representation.

From the table above we see that there is one state with zero minus signs. This is a singlet of $SU(5)$. There are five states with four minus signs. This is a $\overline{5}$ representation of $SU(5)$. Finally there are 10 states with two minus sighs. This is the 10 representation of $SU(5)$. Hence, under $SU(5) \times U(1)_X$ the 16 representation of $SO(10)$ decomposes as:

$$16 = (1, \frac{5}{2}) + (\bar{5}, -\frac{3}{2}) + (10, \frac{1}{2})$$

where the $U(1)_X$ charges are obtained by taking the trace $Q_X = Q_1 + Q_2 + Q_3 + Q_4 + Q_5$, and the $Q_i$ are the $\pm\frac{1}{2}$ charges with respect to the five generators of the Cartan subalgebra.

in our combinatorial notation we can write

$$16 = \binom{5}{0} + \binom{5}{2} + \binom{5}{4}$$

where the combinatorial factor counts the number of $-$ in a given state.

To find the decomposition under $SO(6) \times SO(4) \equiv SU(4) \times SU(2)_L \times SU(2)_R$ we split the five slots into the first three which correspond to $SO(6)$ and the last two which correspond to $SO(4)$. We now split the 16 again by counting how many minus signs there are under the first three times how many there are under the last two. Note that here we have to take the product of the signs with respect to the first three slots and last two separately. So, for example, states with zero or two minus signs under the first three slots belong to the same $SO(6)$ representation. Hence, under $SO(6) \times SO(4) \equiv SU(4) \times SU(2)_L \times SU(2)_R$ it decomposes as:

$$16 = (4, 2, 1) + (\bar{4}, 1, 2)$$

In the combinatorial notation this decomposes as:

$$16 = \left[\binom{3}{0} + \binom{3}{2}\right]\left[\binom{2}{0} + \binom{2}{2}\right] + \left[\binom{3}{1} + \binom{3}{3}\right]\left[\binom{2}{1}\right]$$

To find the decomposition under $SU(3) \times U(1)_C \times SU(2) \times U(1)_L$ we again split the five slots into the first three, which correspond to $SU(3) \times U(1)_C$ and the last two that correspond to $SU(2) \times U(1)_L$. We again count how many minus signs there are in each state under the first three and last two slots to find the multiplicity. The charges under the $U(1)$s are given by the sums under $Q_1 + Q_2 + Q_3$ and $Q_4 + Q_5$ for $U(1)_C$ and $U(1)_L$, respectively. Hence, under $SU(3) \times U(1)_C \times SU(2) \times U(1)_L$ the 16 decomposes as:

$$16 = (1, \frac{3}{2}, 1, +1) + (3, \frac{1}{2}, 2, 0) + (\bar{3}, -\frac{1}{2}, 1, +1) + (1, \frac{3}{2}, 1, -1) + (\bar{3}, -\frac{1}{2}, 1, -1) + (1, -\frac{3}{2}, 1, 0)$$

where $Q_C$ and $Q_L$ are defined as $Q_C = Q_1 + Q_2 + Q_3$ and $Q_L = Q_4 + Q_5$. In combinatorial notation these are:

$$16 = \left[\binom{3}{0}\binom{2}{0}\right] + \left[\binom{3}{2}\binom{2}{0}\right] + \left[\binom{3}{0}\binom{2}{2}\right] + \left[\binom{3}{1}\binom{2}{1}\right] + \left[\binom{3}{2}\binom{2}{2}\right] + \left[\binom{3}{3}\binom{2}{1}\right]$$

2.

a.

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

b.

$$\begin{aligned} t &\rightarrow t + \epsilon A(t,x) \\ x &\rightarrow x + \epsilon B(t,x) \end{aligned}$$

$$\begin{aligned} dt &\rightarrow dt + \epsilon(\tfrac{\partial A}{\partial t} dt + \tfrac{\partial A}{\partial x} dx) \\ dx &\rightarrow dx + \epsilon(\tfrac{\partial B}{\partial t} dt + \tfrac{\partial B}{\partial x} dx) \end{aligned}$$

$$ds^2 \rightarrow [(1 + \epsilon \frac{\partial A}{\partial t})dt + \epsilon \frac{\partial A}{\partial x} dx]^2 - [(1 + \epsilon \frac{\partial B}{\partial x})dx + \epsilon \frac{\partial B}{\partial t} dt]^2$$

we require invariance of $ds^2$. Expanding to first order in $\epsilon$ we impose that the coefficients of the addtional terms vanish. These yield the constraints on the functions $A$ and $B$.

$$\begin{aligned} dt^2 &: \tfrac{\partial A}{\partial t} = 0 \Rightarrow A = A(x) \\ dx^2 &: \tfrac{\partial B}{\partial x} = 0 \Rightarrow B = B(t) \\ dxdt &: \tfrac{\partial A}{\partial x} - \tfrac{\partial B}{\partial t} = 0 \Rightarrow \tfrac{dA}{dx} = \tfrac{dB}{dt} = \text{constant} = c \end{aligned}$$

$$\begin{aligned} \Rightarrow A(x) &= cx + a \\ B(t) &= ct + b \end{aligned}$$

we obtained three constants of integration $a$, $b$ and $c$. These correspond to a shift in time $a$, a shift in space $b$, and a boost $c$.

3.

$$\begin{aligned}
a'^0 &= \gamma(a^0 - \beta a^1) \\
a'^1 &= \gamma(-\beta a^0 + a^1) \\
a'^2 &= a^2 \\
a'^3 &= a^3
\end{aligned}$$

$$\begin{aligned}
b'^0 &= \gamma(b^0 - \beta b^1) \\
b'^1 &= \gamma(-\beta b^0 + b^1) \\
b'^2 &= b^2 \\
b'^3 &= b^3
\end{aligned}$$

$$\begin{aligned}
b'_0 &= -\gamma(b^0 - \beta b^1) \\
b'_1 &= \gamma(-\beta b^0 + b^1) \\
b'_2 &= b^2 \\
b'_3 &= b^3
\end{aligned}$$

Then

$$\begin{aligned}
a'^\mu b'_\mu &= -\gamma(a^0 - \beta a^1)\gamma(b^0 - \beta b^1) + \gamma(-\beta a^0 + a^1)\gamma(-\beta b^0 + b^1) + a^2 b_2 + a^3 b_3 \\
&= \gamma^2 \left[a^0 b^0(-1 + \beta^2) + a^1 b^1(1 - \beta^2)\right] + a^2 b_2 + a^3 b_3 \\
&= -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3 = a^0 b_0 + a^1 b_1 + a^2 b_2 + a^3 b_3 \\
&= a^\mu b_\mu
\end{aligned}$$

4.

Lorentz transformations, derivatives and quantum operators

a.

$$\begin{aligned}
a_0 &= -a^0,\ a_1 = a^1,\ a_2 = a^2,\ a_3 = a^3 \\
a'_0 &= -a'^0 = -\gamma(a^0 - \beta a^1) = \gamma(a_0 + \beta a_1) \\
a'_1 &= a'^1 = \gamma(-\beta a^0 + a^1) = \gamma(\beta a_0 + a_1) \qquad (1) \\
\text{and}\ a'_2 &= a_2,\ a'_3 = a_3
\end{aligned}$$

b.

Suppose we have a function $f(x^0, x^1, x^2, x^3)$ which we express as a function of $x'^0,\ x'^1,\ x'^2,\ x'^3$ by expressing $x^\mu$ as a function of $x'^\mu$. The standard chain rule for partial differentiation says that

$$\frac{\partial f}{\partial x'^\mu} = \sum_{\nu=0}^{3} \frac{\partial f}{\partial x^\nu}\frac{\partial x^\nu}{\partial x'^\mu} \quad \text{for} \quad \mu = 0, 1, 2, 3 \tag{2}$$

Using the summation convention and writing as an operator equation we get

$$\frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} \tag{3}$$

We need $x$ as a function of $x'$, the inverse of the Lorentz transformation that gives $x'$ as a function of $x$. For a boost along the $x'$ axis, the inverse is a boost with the opposite speed, so

$$x^0 = \gamma(x'^0 + \beta x'^1),\ x^1 = \gamma(\beta x'^0 + x'^1) \tag{4}$$

Hence

$$\frac{\partial}{\partial x'^0} = \gamma(\frac{\partial}{\partial x^0} + \beta\frac{\partial}{\partial x^1}), \quad \frac{\partial}{\partial x'^1} = \gamma(\beta\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}) \tag{5}$$

which is the same as as (1) with $a_\mu = \frac{\partial}{\partial x^\mu}$

c.

The operator for momentum $\vec{p}$ is $\frac{\hbar}{i}\vec{\nabla}$, *i.e.*

$$p_1 = \frac{\hbar}{i}\frac{\partial}{\partial x^1},\ p_2 = \frac{\hbar}{i}\frac{\partial}{\partial x^2},\ p_3 = \frac{\hbar}{i}\frac{\partial}{\partial x^3} \tag{6}$$

The Schrödinger equation says

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi$$

where $H$ is the energy operator. Since $p^0 = \frac{E}{c}$ and $x^0 = ct$, this can be written as

$$p^0 = i\hbar\frac{\partial}{\partial x_0} = -\frac{\hbar}{i}\frac{\partial}{\partial x_0} \tag{7}$$

If we write (6) and (7) in terms of $p_\mu$, we remove the sign difference between the 0 component and the others.

$$p_\mu = \frac{\hbar}{i}\frac{\partial}{\partial x^\mu}. \tag{8}$$

This is Lorentz invariant since we proved in (b) that $\frac{\partial}{\partial x^\mu}$ tranforms like $p_\mu$.