MATH 423 midterm November 2011

Full marks can be obtained for complete answers to THREE questions. Only the best THREE answers will be counted.

You may use a university approved pocket calculator and the constants:

$$c = 3 \times 10^{10}\frac{\text{cm}}{s}; \qquad \hbar = 1.054 \times 10^{-27} erg \cdot s;$$

$$G_N = 6.674 \times 10^{-8}\frac{cm^3}{g \cdot s^2} \qquad m_e = 9.109 \times 10^{-28} g;$$

$$m_p = 1.672 \times 10^{-24} g; \qquad k = 1.380 \times 10^{-16}\frac{erg}{K};$$

$$m_{\text{Planck}} = 2.17 \times 10^{-5} g; \qquad m_{\text{sun}} \approx 2 \times 10^{33} g;$$

**1.** (a) Give the Lorentz transformations for the components $a_\mu$ of a vector under a boost along the $x^1$ axis.

[7 marks]

(b) Show that the object $\frac{\partial}{\partial x^\mu}$ transform under a boost along the $x^1$ axis as the $a_\mu$ vector considered in (a) does.

[7 marks]

(c) Show that, in quantum mechanics, the expression for the energy and momentum in terms of derivatives can be written compactly as $p_\mu = \frac{\hbar}{i}\frac{\partial}{\partial x^\mu}$ .

[6 marks]

**2.** (a) Consider the plane $(x, y)$ with the identification

$$(x, y) \sim (x + 2\pi R, y + 2\pi R) \ .$$

What is the resulting space?

[6 marks]

(b) Consider the circle $S^1$, presented as the real line with the identifications $x \sim x + 2$. The circle is the space $-1 < x \leq 1$ with the points $x = \pm 1$ identified. The orbifold $S^1/Z_2$ is defined by imposing the $Z_2$ identification $x \sim -x$. Show that there are two points on the circle that are left fixed by the $Z_2$ action.

[7 marks]

(c) Consider a torus $T^2$, presented as the $(x, y)$ plane with the identifications $x \sim x+2$ and $y \sim y+2$. Choose $-1 < x, y \leq -1$ as the fundamental domain. The orbifold $T^2/Z_2$ is defined by imposing the $Z_2$ identification $(x, y) \sim (-x, -y)$. Prove that there are four points on the torus that are left fixed by the $Z_2$ transformation.

[7 marks]

**3.** (a) The Standard Bohr radius is $a_0 = \frac{\hbar^2}{me^2} \approx 5.29 \times 10^{-9}$cm, and arises from the electric potential $V = -\frac{e^2}{r}$. What would be the gravitational Bohr radius if the attraction force binding the electron to the proton was gravitational?

[9 marks]

(b) In units where $G$, $c$ and $\hbar$ are set equal to one, the temperature of a black hole is given by $kT = \frac{1}{8\pi M}$. Insert back the factors of $G$, $c$ and $\hbar$ into this formula. Evaluate the temperature of a black hole of a million solar masses. What is the mass of a black hole whose temperature is room temperature?

[11 marks]

**4.** A string with tension $T_0$ is stretched from $x = 0$ to $x = 2a$. The part of the string $x \in (0, a)$ has constant mass density $\mu_1$ and the part of the string $x \in (a, 2a)$ has constant mass density $\mu_2$. Consider the differential equation

$$\frac{d^2y}{dx^2} + \frac{\mu(x)}{T_0}\omega^2 y(x) = 0.$$

that determines the normal oscillations.

(a) What boundary conditions should be imposed on $y(x)$ and $\frac{dy}{dx}(x)$ at $x = a$ ?

[5 marks]

(b) Write the conditions that determine the possible frequencies of oscillation.

[15 marks]