LST. 97 | 13/11/09. / / Abingdon / Friday /

For the time being:

we accept that the ordering of L_0 is ambiguous and write the constraints in the form.

$$\langle \phi | L_0 - a | \phi \rangle = 0 \qquad \langle \phi | \tilde{L}_0 - \tilde{a} | \phi \rangle = 0$$

where L_0 $\tilde{L}_0$ are normally ordered, and $a, \tilde{a}$ are constants.

It turns out that:

$$\langle \phi | \phi \rangle \geq 0 \text{ for all } |\phi\rangle \Rightarrow a, \tilde{a} = 1$$

→ ambiguity is completely fixed by physical requirements,

→ physical states satisfy:

$$\langle \phi | L_m | \phi \rangle = 0 \qquad \langle \phi | \tilde{L}_m | \phi \rangle = 0 \quad \text{for } m \neq 0$$

$$\langle \phi | L_0 - a | \phi \rangle = 0 \qquad \langle \phi | \tilde{L}_0 - \tilde{a} | \phi \rangle = 0$$

As we saw: classically L_0 and $\tilde{L}_0$ give an expression for the mass of the string.

$$\alpha_0 = \frac{1}{2} l_s p^\mu \qquad \alpha' = \frac{1}{2} l_s^2$$

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n = \frac{1}{4}\alpha' p^2 + \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n$$

and similarly for $\tilde{L}_0$ in the case of the closed string.

Hence classically: $M^2 = \frac{4}{\alpha'}\left(\sum_{n=1}^{\infty} \alpha_{-n}\alpha_n\right) = \frac{4}{\alpha'}\left(\sum \tilde{\alpha}_{-n}\tilde{\alpha}_n\right)$

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Quantum mechanically

This is modified due to the normal ordering factor a, $\tilde{a}$.

analysis | of Physical States

Hermiticity of $L_m \Rightarrow L_m^{+} = L_{-m}$.

$\Rightarrow$ write the constraint in the form.

$L_m|\phi\rangle = 0 \quad \tilde{L}_m|\phi\rangle = 0 \quad m > 0.$

$(L_0 - a)|\phi\rangle = 0 \quad (\tilde{L}_0 - \tilde{a})|\phi\rangle = 0$

From the L_0 constraint.

$(L_0 - a)|\phi\rangle = 0$

$\Rightarrow (\frac{1}{4}\alpha' p^2 + \hat{N} - a)\;|\phi\rangle = 0.$ $\quad \hat{N} = \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n$

number operator

$\Rightarrow \frac{\alpha'}{4} M^2 = -a + \hat{N}$

and From $(\tilde{L}_0 - \tilde{a})|\phi\rangle = 0$

$\frac{\alpha'}{4} M^2 = -\tilde{a} + \hat{\tilde{N}}$

The total mass of the string is obtained from.

$H = 2(L_0 + \tilde{L}_0). \Rightarrow \frac{\alpha'}{2} M^2 = \hat{N} + \hat{\tilde{N}} - a - \tilde{a}$

and level matching requires $(L_0 - a - \tilde{L}_0 - \tilde{a})|\phi\rangle = 0$

For the bosonic string $a = \tilde{a} \Rightarrow \hat{N} = \hat{\tilde{N}}$.

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For the open string

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n = \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n + \alpha' p^2.$$

$$\Rightarrow M^2 = \frac{1}{\alpha'} \sum_{n=} \alpha_{-n}\alpha_n \rightarrow \text{classically.}$$

and $(L_0 - a)|\phi\rangle = 0 \Rightarrow \alpha' M^2 = N - a$ Quantumally.

<u>what are the lowest mass states?</u>

open string | $a = 1$.

N	$\alpha' M^2$	state	spin
0	-1	$\|k\rangle$	scalar.
1	0	$\alpha^{\mu}_{-1}\|k\rangle$	vector.
2	1	$\alpha^{\mu}_{-2}\|1\rangle$	vector.
		$\alpha^{\mu}_{-1}\alpha^{\nu}_{-1}\|k\rangle$	symmetric tensor.

we still have to impose the constraints $L_m|\phi\rangle = 0 \quad m > 0$

For the state $|\phi\rangle = |k\rangle$ it is obvious that $L_m|k\rangle = L_m e^{-ikx}|0\rangle = 0$ for $m > 0$.

Hence, the state $|\phi\rangle = |k\rangle$ is physical.

→ Lowest mass eigenstate → ground state of the Relativistic string.

→ Ground state is tachyonic $M^2 < 0$.

→ Propagates faster than the speed of light → unphysical.

→ Fermionic string → space-time supersymmetry → No tachyon.

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The massless vector states (Photon) of the open string

we take $a=1 \rightarrow$ not yet proven.

The constraint $L_1|\phi\rangle$ is non-trivial.

$L_1|\phi\rangle = L_1 \rho_\mu \alpha^\mu_{-1}|k\rangle = 0$.

with $L_1 = -\frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{1-n}\alpha_n \Rightarrow$ only $\alpha_1 \cdot \alpha_0$ non-trivial

$\Rightarrow \alpha_{1\nu}\alpha_0^\nu \rho_\mu \alpha^\mu_{-1}|k\rangle = \alpha_0^\nu \rho_\mu (\alpha_{1\nu}\alpha^\mu_{-1} - \alpha^\mu_{-1}\alpha_{1\nu} + \alpha^\mu_{-1}\alpha_{1\nu})|k\rangle$

$= \alpha_0^\nu \rho_\mu [\alpha_{1\nu}, \alpha^\mu_{-1}]|k\rangle = \alpha_0^\nu \rho^\mu [\underbrace{\alpha_{1\nu}, \alpha_{-1\mu}}_{m\delta_{m+n}\eta_{\mu\nu}}]|k\rangle = \rho^\mu \alpha_{0\mu}|k\rangle$

$\alpha_{0\mu} = \hat{P}_\mu$ open string with $l_s=1$

$= \rho^\mu k_\mu |0\rangle = 0$.

$\Rightarrow$ The physical vector state satisfies.

1) $M^2 = -k^2 = -k_\mu k^\mu = 0 \rightarrow$ massless.

2) $k^\mu \rho_\mu = 0$. $\rho^\mu \rightarrow$ polarization vector.

FOR $m > 1 \Rightarrow L_m \rho_\mu \alpha^\mu_{-1}|k\rangle = 0$.

$\rightarrow$ The vector states subject to the conditions 1) and 2) are physical.

can we intepprete these states a components of a photon?

In D dimensions the photon has D-2 independent components, due to gauge invariance.

Recall | the action of electromagnetic field with charged matter

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$$S[A,j] = \int d^D x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu\right).$$

→ Maxwell eq. $\partial^\mu F_{\mu\nu} = j_\nu$.

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

The field the field strength and the action are invariant under gauge transformations $A_\mu \to A_\mu + \partial_\mu \chi$.

Consider the free Maxwell theory with $j^\mu = 0$.

→ $\partial^\mu F_{\mu\nu} = \partial^\mu \partial_\mu A_\nu - \partial^\mu \partial_\nu A_\mu = \Box A_\nu - \partial_\nu \partial^\mu A_\mu = 0$.

Lorentz gauge: $\partial^\mu A_\mu = 0 \Rightarrow \Box A_\nu = 0$.

There is still a residual gauge symmetry because

set: $\Box \chi = 0 \Rightarrow \partial^\nu A_\nu \to \partial^\nu (A_\nu + \partial_\nu \chi) = \partial^\nu A_\nu + \Box \chi$.

→ the states A_μ and $A_\mu + \partial_\mu \chi$ with $\Box \chi = 0$

Represent the same physical state →

→ only $D-2$ independent components

consider. $A_\mu = f_\mu e^{ik\cdot x}$ → f_μ polarization vector.
k_μ momentum vector.

$\Box A_\mu = 0 \iff k_\mu k^\mu = 0$.

$\partial^\mu A_\mu = 0 \iff f^\mu k_\mu = 0$.

$A_\mu \simeq A_\mu + \partial_\mu \chi \iff f_\mu \simeq f_\mu + \lambda k_\mu$ o.k.

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Tache |

$k = (k^0, 0, \ldots, 0, k^0)$.

corresponding to propogation of a massless particle along D-1 ax

$k \cdot \rho = 0 \Rightarrow -k^0\rho^0 + k^0\rho^{(D-1)} = 0 \Rightarrow \rho^0 = \rho^{(D-1)}$

$\Rightarrow$ $\rho = (\rho^0, \rho^1 \cdots \rho^{D-2}, \rho^{D-1}) \rightarrow$ physical polarization vector.

$\rightarrow$ introduce the following basis for polarization vectors.

$k = (k^0, \ldots, k^0)$.

$\bar{k} = \frac{1}{2(k^0)^2}(-k^0, \ldots, k^0)$.

$e_i = (0, \ldots, 1, \ldots, 0)$ $\qquad i = 1, \ldots, D-2$

$\rightarrow$ orthonormal basis with.

$k \cdot k = 0 \qquad k \cdot \bar{k} = 1 \qquad \bar{k} \cdot \bar{k} = 0 \qquad e_i \cdot e_j = 0. \; e_i \cdot e_i =$

$k \cdot e_i = 0 \qquad \bar{k} \cdot e_i = 0$.

$\rightarrow$ $k, \bar{k}$ - lightlike vectors spanning the light-cone.

e_i - D-2 vectors spanning the trans-verse space.

general physical polarization: $\rho = \rho^i e_i + \alpha k$.

$\rightarrow$ unphysical polarization $\rightarrow$ parallel to $\bar{k}$ (sinke $k \cdot \bar{k} \neq 0$).

However, the part of ρ which is parallel to k.

is also unphysical $\rightarrow$ it has zero norm $k^2 = 0$.

2) it is orthogonal to any physical polarization vector.

$\Rightarrow$ The D-2 physical degrees of freedom are in the transverse part: $\rho_{\text{transverse}} = \rho^i \epsilon_i = (0, \rho^1, \rho^2, \dots, \rho^{D-2}, 0)$

$\rightarrow$ Transverse: because the spatial part:

$$\vec{\rho} = (\rho^1, \rho^2, \dots, \rho^{D-2}, 0).$$

is orthogonal to the spatial momentum.

$$\vec{k} = (0, \dots, 0, k^0)$$

In $D=4 \rightarrow$ Photon has two polarizations transverse to the momentum.

Returning to the case of the open string:

we saw that the physical states $(\rho_\mu \alpha_{-1}^\mu |k\rangle)$ satisfy.

$$k^2 = 0 \qquad k^\mu \rho_\mu = 0.$$

consider the states of the form:

$$|\psi\rangle = \lambda k_\mu \alpha_{-1}{}^\mu |k\rangle$$

where λ is a real constant.

take | $\langle\psi|\psi\rangle = \lambda^2 k_\mu k_\nu \langle k| [\alpha_1{}^\nu, \alpha_{-1}{}^\mu] |k\rangle = \lambda^2 k^2 \langle k|k\rangle = 0$

as $k^2 = 0$.

The state $|\psi\rangle \neq 0$ yet $\langle\psi|\psi\rangle = 0$.

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→ $|\psi\rangle$ is called a null state.

→ There is a residual gauge symmetry that has to be imposed on the physical spectrum.

Further | $|\psi\rangle$ is orthogonal to all states.

For level 1 states: $\langle k'|\alpha_1^{\mu}\, \rho_\mu\, k_\nu \alpha_{-1}^{\nu}|k\rangle = \rho\cdot k\,\langle k'|k\rangle = 0$

because either $k \neq k' \Rightarrow \langle k'|k\rangle = 0$.

or $k = k' \Rightarrow \rho\cdot k = 0$.

⇒ $|\psi\rangle$ is a 'spurios state' because it drops out from the scalar product with any physical state.

i.e. $(\rho_\mu + \lambda k_\mu)\,\alpha_{-1}^{\mu}|k\rangle$.

Represents the same physical state for any value of λ → corresponding to gauge degree of freedom.

→ The component of ρ parallel to the momentum is unphysical

→ D-2 independent polarizations.

→ The massless vector state of the open string has the kinetic properties that characterize a photon

i.e. a massless vector boson with a gauge invariance.

→ Maxwell action: space-time action for the massless vector state of the open string.

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States of the closed string

Physical close string states satisfy the mass-shell and level matching conditions:

$$\alpha' M^2 = 2(N + \tilde{N} - a - \tilde{a})$$

$$N = \tilde{N}$$

and $L_m |\phi\rangle = 0 \quad \tilde{L}_m |\phi\rangle = 0 \quad$ for $m > 0$.

Taking $a = \tilde{a} = 1$.

The resulting states are :

$N = \tilde{N}$	$\alpha' M^2$	state	spin
0	-4	$\|k\rangle$	scalar
1	0	$\alpha^\mu_{-1} \tilde{\alpha}^\nu_{-1} \|k\rangle$	2nd Rank tensor
2	4	$\alpha^\mu_{-1} \alpha^\nu_{-1} \tilde{\alpha}^\rho_{-1} \tilde{\alpha}^\sigma_{-1} \|k\rangle$	4th Rank tensor
		$\alpha^\mu_{-1} \alpha^\nu_{-1} \tilde{\alpha}^\rho_{-2} \|k\rangle$	3rd Rank tensor
		$\alpha^\mu_{-2} \tilde{\alpha}^\rho_{-1} \tilde{\alpha}^\sigma_{-1} \|k\rangle$	3rd Rank tensor
		$\alpha^\mu_{-2} \tilde{\alpha}^\rho_{-2} \|k\rangle$	2nd Rank tensor

The ground state is again a tachyonic scalar.

The tachyon is eliminated in the fermionic string with space-time supersymmetry

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The first excited state is massless, (for $a = \tilde{a} = 1$)

and a second rank tensor.

$$\rho_{\mu\nu}\, \alpha_{-1}^{\mu} \tilde{\alpha}_{-1}^{\nu} |k\rangle \rightarrow$$ Linear combination of level-two states.

<u>Constraints:</u> $L_1(\rho_{\mu\nu}\alpha_{-1}^{\mu}\tilde{\alpha}_{-1}^{\nu}|k\rangle) = \tilde{L}_1(\quad) = 0$

<u>Evaluate</u> $L_1 \rho_{\mu\nu}\alpha_{-1}^{\mu}\tilde{\alpha}_{-1}^{\nu}|k\rangle = \left(\frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{1-n}^{\alpha}\alpha_{n\alpha}\right)\rho_{\mu\nu}\alpha_{-1}^{\mu}\tilde{\alpha}_{-1}^{\nu}|k\rangle =$

$= \alpha_0^{\alpha}\alpha_{1\alpha}\rho_{\mu\nu}\alpha_{-1}^{\mu}\tilde{\alpha}_{-1}^{\nu}|k\rangle =$

$= \alpha_0^{\alpha}\rho_{\mu\nu}(\alpha_{1\alpha}\alpha_{-1}^{\mu} - \alpha_{-1}^{\mu}\alpha_{1\alpha} + \alpha_{-1}^{\mu}\alpha_{1\alpha})\tilde{\alpha}_{-1}^{\nu}|k\rangle$

$= \alpha_0^{\alpha}\rho_{\mu\nu}[\alpha_{1\alpha}, \alpha_{-1}^{\mu}]\tilde{\alpha}_{-1}^{\nu}|k\rangle = \alpha_0^{\alpha}\rho_{\mu\nu}\eta_{\alpha}^{\mu}\tilde{\alpha}_{-1}^{\nu}|k\rangle$

$\boxed{\alpha_0^{\alpha} = \frac{1}{2}l_s p^{\alpha}}$ $= \frac{l_s}{2}\rho_{\mu\nu}\eta_{\alpha}^{\mu}k^{\alpha}\tilde{\alpha}_{-1}^{\nu}|k\rangle = 0.$

$\Rightarrow \eta^{\alpha\mu}\rho_{\mu\nu}k_{\alpha} = 0$ OR $k^{\alpha}\rho_{\alpha\nu} = 0.$

<u>similarly:</u> $\tilde{L}_1(\rho_{\mu\nu}\alpha_{-1}^{\mu}\tilde{\alpha}_{-1}^{\nu}|k\rangle) = 0 \Rightarrow \rho_{\mu\alpha}k^{\alpha} = 0.$

and $k^2 = -M^2 = 0.$

$\rho_{\mu\nu} \rightarrow 2^{nd}$ Rank tensor, under the $SO(1, D-1)$ Lorentz group.

$\rightarrow \rho_{\mu\nu} \rightarrow$ reducible representation of the Lorentz group

$\rho_{\mu\nu} = S_{\mu\nu} + b_{\mu\nu}$

where $S_{\mu\nu} = \rho_{(\mu\nu)} = \frac{1}{2}(\rho_{\mu\nu} + \rho_{\nu\mu}).$

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and $b_{\mu\nu} = \rho_{[\mu\nu]} = \frac{1}{2}(\rho_{\mu\nu} - \rho_{\nu\mu})$.

The polarization tensors satisfy the constraints

$k^\mu s_{\mu\nu} = 0$. $k^\mu b_{\mu\nu} = 0$.

The antisymmetric part satisfies $b_{\mu\nu} = -b_{\nu\mu}$.

$k^2 = 0 \quad k^\mu b_{\mu\nu} = 0$.

It gives rise to a massless rank 2 antisymmetric gauge field → Plays an important role in phenomenology.

→ axion field in 4D.

as well as in theory → Generalised Geometry (Hitchin).

The symmetric part satisfies:

$s_{\mu\nu} = s_{\nu\mu}$; $k^2 = 0 \quad k^\mu s_{\mu\nu} = 0$.

In contrast to the antisymmetric tensor $b_{\mu\nu}$, $s_{\mu\nu}$ is reducible. be

→ traceless part: $\psi_{\mu\nu} = s_{\mu\nu} - \frac{1}{(D-2)} s^\rho{}_\rho (\eta_{\mu\nu} - k_\mu \bar{k}_\nu - k_\nu \bar{k}_\mu)$.

where $k^2 = 0 \quad \bar{k}^2 = 0 \quad k^\mu \bar{k}_\mu = 1$.

k^μ - momentum vector, $\bar{k}$ - another lightlike vector.

This definition ensures that $\psi_{\mu\nu}$ is physical i.e. not spurios.

and $s^\rho{}_\rho = \eta^{\mu\nu} s_{\mu\nu} \neq 0$ but $\eta^{\mu\nu} \psi_{\mu\nu} = 0$.

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$\psi_{\mu\nu} \rightarrow$ graviton.

The remaining pure trace part of $S_{\mu\nu}$ is

$$\phi_{\mu\nu} = \frac{1}{(D-2)} S^{\rho}{}_{\rho} \left(\eta_{\mu\nu} - k_\mu \bar{k}_\nu - k_\nu \bar{k}_\mu \right).$$

<u>Note that:</u> $S_{\mu\nu} = \psi_{\mu\nu} + \phi_{\mu\nu}$

$$\eta^{\mu\nu} \phi_{\mu\nu} = S^{\rho}{}_{\rho}.$$

The trace part satisfies $k^\mu \phi_{\mu\nu} = 0$; $k^2 = 0$

it is physical, i.e. not spurios.

$\phi_{\mu\nu} \rightarrow$ Dilaton.

similar to the case of the photon we can work out the physical and spurios components of the graviton / dilaton and antisymmetric tensor fields.

similar to the photon case we can remove the unphysical states by using the gauge symmetry

$b_{\mu\nu} \rightarrow b_{\mu\nu} + k_\mu a_\nu - k_\nu a_\mu$ where $k^\mu a_\mu = 0$

and

$S_{\mu\nu} \rightarrow S_{\mu\nu} + k_\mu f_\nu + k_\nu f_\mu$ where $k^\mu f_\mu = 0$

Light-cone quantization.

we saw that:

@ imposing canonical commutation relations:

we gain. → manifest Lorentz covariance. (invariance).

we lose → unphysical states, Ghosts, spurios states.

An alternative quantization scheme → light cone.

→ Pick a special set of light-cone coordinates

we lose → manifest Lorentz invariance.

we gain → only physical states.

However: 1) a ghost free spectrum is only possible

with $D = 26$ $a = 1$ $\tilde{a} = 1$

conversely: 2) Lorentz invariance is only possible with.

$D = 26$ $a = 1$ $\tilde{a} = 1$.

Light-cone gauge → Useful to extract the physical spectrum and dynamics.

We start by reviewing light-cone coordinates in special relativity. We will then specialize to the relativistic string.

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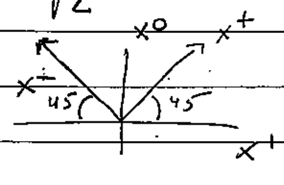
Light-cone coordinates in special relativity

consider: $x^\mu = (x^0, x^1, x^2, x^3)$

Define | $x^+ = \frac{1}{\sqrt{2}}(x^0 + x^1)$ $\quad x^- = \frac{1}{\sqrt{2}}(x^0 - x^1)$

trade : $x^0, x^1 \longrightarrow x^+, x^-$.



$x^\mu \longrightarrow (x^+, x^-, x^2, x^3)$.

we have $-ds^2 = -\eta_{\mu\nu} dx^\mu dx^\nu = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$

$$= -(dx^0 + dx^1)(dx^0 - dx^1) + (dx^2)^2 + (dx^3)^2 = -2dx^+ dx^- + (dx^2)^2 + (dx^3)^2$$

$$= \hat{\eta}_{\mu\nu} dx^\mu dx^\nu = -2dx^+ dx^- + (dx^2)^2 + (dx^3)^2.$$

with

$$\hat{\eta}_{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

any vector $a^+ = \frac{1}{\sqrt{2}}(a^0 + a^1)$ $\quad a^- = \frac{1}{\sqrt{2}}(a^0 - a^1)$

$$a \cdot b = -a^- b^+ - a^+ b^- + a^2 b^2 + a^3 b^3 = \hat{\eta}_{\mu\nu} a^\mu b^\nu.$$

write $a \cdot b = a_\mu b^\mu = a_+ b^+ + a_- b^- + a_1 b^1 + a_2 b^2$

and $a_+ = -a^-$ $\quad a_- = -a^+$

the indices switch plus a change of sign.

Example | consider a particle moving along the x^1 axis with speed $\beta = v/c$. At time $t = 0$ the positions x^1, x^2 and x^3 are all zero.

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Then | $X^1(t) = vt = \beta X^0 \qquad X^2(t) = X^3(t) = 0.$

How does this look in light-cone coordinates?

$$X^+ = \frac{X^0 + X^1}{\sqrt{2}} = \frac{1+\beta}{\sqrt{2}} X^0$$

$$X^- = \frac{X^0 - X^1}{\sqrt{2}} = \frac{1-\beta}{\sqrt{2}} X^0 = \frac{1-\beta}{1+\beta} X^+$$

$$\Rightarrow \frac{dX^-}{dX^+} = \frac{1-\beta}{1+\beta}$$

we can think of X^+ as a new "time" coordinate

$\Rightarrow \frac{dx^-}{dx^+} \longrightarrow$ light cone velocity.

Light-cone coordinates $\rightarrow$ not a Lorentz frame.

no $\Lambda^\mu{}_\nu$ | such that: $X'^\mu = \Lambda^\mu{}_\nu X^\nu$
and $X'^\mu = (X^+, X^-, X^2, X^3)$
where X^ν is a Lorentz frame.

Light-cone energy and momentum

using our earlier rule: $p^+ = \frac{1}{\sqrt{2}}(p^0 + p^1) = -p_-$

$$p^- = \frac{1}{\sqrt{2}}(p^0 - p^1) = -p_+$$

since: $p^0 = \frac{E}{c} = \sqrt{\vec{p}\cdot\vec{p} + m^2c^2} > |\vec{p}| \geq |p^1|$

both $p^\pm = p^0 \pm p^1 > 0.$

consider $p\cdot X = p_0 X^0 + p_1 X^1 + p_2 X^2 + p_3 X^3$

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In Light-coone coordinates we have

(*) $P \cdot X = P_+ X^+ + P_- X^- + P_2 X^2 + P_3 X^3$

we identified X^+ as our new "time" coordinate.

In ordinary coordinates: $i\hbar \frac{\partial}{\partial X^0} \psi = \frac{E}{c} \psi$

and $\psi = e^{-\frac{i}{\hbar}(Et - \vec{p}\cdot\vec{x})}$

how should we write it in L.C.C. ? $i\hbar \frac{\partial}{\partial X^+} \psi = \frac{E_{lc}}{c}$?

From (*) $\psi(x) = e^{\frac{i}{\hbar}(p_0 X^0 + \vec{p}\cdot\vec{x})} = e^{ip\cdot x}$

$= e^{\frac{i}{\hbar}(P_+ X^+ + P_- X^- + P_2 X^2 + P_3 X^3)}$

$\Rightarrow i\hbar \frac{\partial \psi}{\partial X^+} = -P_+ \psi \rightarrow -P_+ = \frac{E_{lc}}{c}$

$\rightarrow$ identify $P^- = -P_+ = \frac{E_{lc}}{c}$

as the Light-cone energy.

Light-cone Relativistic string

we proceed to study the relativistic string in light cone coordinates.

Light-cone coordinates in target space.

Completely fixes the reparameterization and Weyl rescaling freedom

$\rightarrow$ i.e. $\rightarrow$ implements the constraints $\rightarrow$ physical spectrum

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Recall : conformal gauge $h_{\alpha\beta} = \eta_{\alpha\beta}$.

However Remaining gauge freedom : $\eta_{\alpha\beta} \to \Lambda^2(\sigma)\,\eta_{\alpha\beta}$.

can be undone by a Weyl transformation.

→ make an additional gauge choice that completely fixes the constraint

what are the world-sheet coordinates ?

Use : Light-cone world sheet coordinates : $\sigma^{\pm} = \tau \pm \sigma$.

with $ds^2 = -d\sigma^+ d\sigma^-$

→ any transformation of the form

$$\sigma^+ \to \tilde{\sigma}^+(\sigma^+) \quad , \quad \sigma^- \to \tilde{\sigma}^-(\sigma^-)$$

multiplies the flat metric by an overall constant.

→ Can be undone by a compensating Weyl transformation.

comments : 1) The remaining gauge freedom ↔ $\tilde{\sigma}^+(\sigma^+)$ $\tilde{\sigma}^-(\sigma^-)$

2) we start with $X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-)$ → $2D$ functions.

constraints $(\partial_+ X)^2 = (\partial_- X)^2 = 0$ → $2D-2$ functions

However the freedom $\tilde{\sigma}^+(\sigma^+)$ $\tilde{\sigma}^-(\sigma^-)$ → $2D-4$ functions.

Interpretation → transverse fluctuation of the string.

3) Recall static gauge $X^0 = R\tau$ → not sufficient

→ Light cone gauge.

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Light-cone gauge: introduce Light-cone coordinates in space-time

$$X^{\pm} = \frac{1}{\sqrt{2}}\left(X^0 \pm X^{D-1} \right)$$

and X^i $\quad i = 1, \dots, D-2$ are unchanged.

$\Rightarrow$ Formulation is not covariant $\xrightarrow{Q.M.}$ Anomalies $\Rightarrow D=26; \tilde{a}=1$.

The space time Minkowski metric: $ds^2 = -2dX^+dX^- + \sum_{i=1}^{D-2} dX^i dX^i$

$\Rightarrow$ The inner product of two arbitrary vector:

$$v \cdot w = v_\mu w^\mu = -v^+w^- - v^-w^+ + \sum_i v^i w^i.$$

Indices are raised and lowered by the rule

$$v^- = -v_+ \quad v^+ = -v_- \qquad v^i = v_i.$$

In terms of the $\tilde{\sigma}^{\pm}(\sigma^{\pm})$ we have

$$\tilde{\tau} = \frac{1}{2}\left[\tilde{\sigma}^+(\sigma^+) + \tilde{\sigma}^-(\sigma^-) \right]$$

$$\tilde{\sigma} = \frac{1}{2}\left[\tilde{\sigma}^+(\sigma^+) - \tilde{\sigma}^-(\sigma^-) \right]$$

This means that $\tilde{\tau}$ can be an arbitrary solution of the wave equation

$$\left(\frac{\partial^2}{\partial \sigma^2} - \frac{\partial^2}{\partial \tau^2} \right) \tilde{\tau} = 0.$$

once $\tilde{\tau}$ is determined $\tilde{\sigma}$ is specified up to a constant.

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Recall $X^+ = X_L^+(\sigma^+) + X_R^+(\sigma^-)$.

gauge fix $\longrightarrow$ $X_L^+ = \frac{1}{2}x^+ + \frac{1}{2}\alpha' p^+ \sigma^+$

$X_R^+ = \frac{1}{2}x^+ + \frac{1}{2}\alpha' p^+ \sigma^-$

and $X^+ = x^+ + \alpha' p^+ \tau$.

$\longrightarrow$ Light-cone gauge $\rightarrow \alpha_n^+ = 0$ for $n \neq 0$

solving for X^-

X^- satisfies the wave eq. $\partial_+ \partial_- X^- = 0$

which is solved by

$$X^- = X_L^-(\sigma^+) + X_R^-(\sigma^-)$$

The constraints : $(\partial_+ X)^2 = (\partial_- X)^2 = 0$.

in L.C.C. $\longrightarrow$ $\partial X_+^- \partial_+ X^+ = \sum_{i=1}^{2} \partial_+ X^i \partial_+ X^i$

Since $\partial_+ X^+ = \alpha' p^+$

$$\Rightarrow \quad \partial_+ X_L^- = \frac{1}{\alpha' p^+} \sum_{i=1}^{D-2} \partial_+ X^i \partial_+ X^i$$

similarly. $$\partial_- X_R^- = \frac{1}{\alpha' p^+} \sum_{i=1}^{D-2} \partial_- X^i \partial_- X^i$$

$\Rightarrow$ up to an integration constant $X^-(\sigma^+, \sigma^-)$ is determined by the transverse fields X^i with $i = 1, \dots, D-2$.

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$\Rightarrow X_L^-(\sigma^+) = \frac{1}{2}x^- + \frac{1}{2}\alpha' p^- \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \tilde{\alpha}_n^- e^{-in\sigma^+}$

$X_R^-(\sigma^-) = \frac{1}{2}x^- + \frac{1}{2}\alpha' p^- \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \alpha_n^- e^{-in\sigma^-}$

$\rightarrow$ x^- is an integration constant.

$p^-, \alpha_n, \tilde{\alpha}_n$ are fixed by the constraints

e.g. $\alpha_n^- = \frac{1}{\sqrt{2\alpha'}} \frac{1}{p^+} \sum_{i=1}^{D-2} \sum_{m=-\infty}^{\infty} \alpha_{n-m}^i \alpha_m^i$

a special case is for $n=0 \rightarrow \alpha_0^- = \sqrt{\alpha'/2}\, p^-$

$$\Rightarrow \frac{\alpha' p^-}{2} = \frac{1}{2p^+}\left(\sum_{i=1}^{D-2} \frac{1}{2}\alpha' p^i p^i + \sum_{n\neq 0} \alpha_n^i \alpha_{-n}^i\right)$$

we get a similar equation from the constraints $(\partial_+ X)^2 = 0$

$$\Rightarrow \frac{\alpha' p^-}{2} = \frac{1}{2p^+}\left(\sum_{i=1}^{D-2} \frac{1}{2}\alpha' p^i p^i + \sum_{n\neq 0} \tilde{\alpha}_n^i \tilde{\alpha}_{-n}^i\right)$$

The classical level conditions $L_n = \tilde{L}_n = 0 \quad n \in \mathbb{Z}$ give:

$$M^2 = 2p^+p^- - \sum_{i=1}^{D-2} p^i p^i = \frac{4}{\alpha'} \sum_{i=1}^{D-2} \sum_{n>0} \alpha_{-n}^i \alpha_n^i = \frac{4}{\alpha'} \sum_{i=1}^{D-2} \sum_{n>0} \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i$$

The sum is only over the transverse oscillators.

α^i and $\tilde{\alpha}^i \qquad i = 1, \ldots, D-2.$

$\rightarrow$ in the light-cone gauge only the transverse oscillations are independent.

$\rightarrow$ classical solution: $\alpha_n^i\ \tilde{\alpha}_n^i,\ \underbrace{x^i, p^i, p^+, x^+}_{\text{c.o.m. position and momentum.}},\ x^- \qquad i = 1,\ldots,D-2$

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Quantization

so far our discussion of the L.C.S was classical.

Next: impose commutation relations:

$$[X^i, P^j] = i\delta^{ij} \qquad [X^-, P^+] = -i$$

$$[\alpha_n^i, \alpha_m^j] = [\tilde{\alpha}_n^i, \tilde{\alpha}_m^j] = n\delta^{ij}\delta_{n+m,0}$$

→ similar to those in the covariant quantization approach.

However, we are missing $[X^+, ?] = ?$

since P^- is not an independent dynamical variable

→ It is fixed by the constraints.

Proceed by Imposing $[X^+, P^-] = -i$ plus constraints on P^-.

Define a vacuum state such that.

$$\hat{P}^\mu |0;P\rangle = p^\mu |0,P\rangle$$

$$\alpha_n^i |0;P\rangle = \tilde{\alpha}_n^i |0,P\rangle = 0 \quad \text{for } n > 0.$$

but now $i = 1, \dots, D-2$ → only transverse modes.

→ Hilbert space is positive definite → no ghosts.

Constraints

Imposing the constraint on P^- is equivalent to imposing the mass shell condition as an operator equation.

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Like the covariant quantization case → normal ordering ambiguity

$$\Rightarrow M^2 = \frac{4}{\alpha'}\left(\sum_{i=1}^{D-2}\sum_{n>0}\alpha_{-n}^i\alpha_n^i - a\right) = \frac{4}{\alpha'}\left(\sum_{i=1}^{D-2}\sum_{n>0}\tilde{\alpha}_{-n}^i\tilde{\alpha}_n - \tilde{a}\right)$$

$$= \frac{4}{\alpha'}(N-a) = \frac{4}{\alpha'}(\tilde{N}-\tilde{a}) \quad \text{with } a=\tilde{a}$$

and

open string: $M^2 = \frac{1}{\alpha'}(N-a) \qquad N = \sum_{i=1}^{D-2}\sum_{n=1}^{\infty}\alpha_{-n}^i\alpha_n^i$

In the light-cone gauge all excitations are generated by acting with α_{-n}^i ; $i=1,\dots,D-2$.

→ First excited state $\alpha_{-1}^i|0;P\rangle \qquad i=1,\dots,D-2$.

→ $D-2$ components of space time rep of $SO(D-2)$ Rotation group in transverse space.

Lorentz invariance | massive: Reps of $SO(D-1)$.

massless: Reps of $SO(D-2)$.

Hence: Lorentz invariance dictates that $a=1$.

Determining D.

Recall: From L_0

$$\frac{1}{2}\sum_{n\neq 0}\alpha_{-n}^i\alpha_n^i = \frac{1}{2}\sum_{n<0}\alpha_{-n}\alpha_n^i + \frac{1}{2}\sum_{n>0}\alpha_{-n}^i\alpha_n^i$$

$$= \frac{1}{2}\sum_{n<0}\left[\alpha_n^i\alpha_{-n}^i - n(D-2)\right] + \frac{1}{2}\sum_{n>0}\alpha_{-n}^i\alpha_n^i$$

$$= \sum_{n>0}\alpha_{-n}^i\alpha_n^i + \frac{D-2}{2}\sum_{n>0}n.$$

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we need to evaluate $\sum_{n=1}^{\infty} n$. and renormalize.

$$\sum_{n=1}^{\infty} n \rightarrow \sum_{n=1}^{\infty} n e^{-n\epsilon} \qquad \epsilon << 1$$

$$= -\frac{\partial}{\partial\epsilon} \sum_{n=1}^{\infty} e^{-n\epsilon} = -\frac{\partial}{\partial\epsilon} \frac{1}{(1-e^{-\epsilon})}$$

$$= \frac{e^{-\epsilon}}{(1-e^{-\epsilon})^2} = \frac{1}{e^{\epsilon}(1-e^{-\epsilon})^2} =$$

$$= \frac{1}{e^{\epsilon}(1-2e^{-\epsilon}+e^{-2\epsilon})} = \frac{1}{(e^{\epsilon}-2+e^{-\epsilon})}$$

$$= \frac{1}{\left(1+\epsilon+\frac{\epsilon^2}{2}+\frac{\epsilon^3}{6}+\frac{\epsilon^4}{24}-2+1-\epsilon+\frac{\epsilon^2}{2}-\frac{\epsilon^3}{6}+\frac{\epsilon^4}{24}+\dots\right)}$$

$$= \frac{1}{\epsilon^2+\frac{\epsilon^4}{12}} = \frac{1}{\epsilon^2}\frac{1}{\left(1+\frac{\epsilon^2}{12}\right)} = \frac{1}{\epsilon^2}\left(1-\frac{\epsilon^2}{12}\right) = \frac{1}{\epsilon^2} - \frac{1}{12}.$$

$\frac{1}{\epsilon^2} \rightarrow \infty \rightarrow$ renormalize.

Finite term of: $\sum_{n=1}^{\infty} n = -\frac{1}{12} + \lim_{\epsilon\to 0} = -\frac{1}{12}.$

$$\Rightarrow a = \frac{D-2}{24} = 1 \Rightarrow D = 26.$$

$\Rightarrow$ Lorentz invariance is possible only for $a=1$ $D=26$.