LST, 73 | 18/10/09.7 | Abingdon | sunday |

The canonical momenta is given by

$$\Pi^\mu = \frac{\partial \mathcal{L}_P}{\partial \dot{X}_\mu} = \frac{\partial}{\partial \dot{X}_\mu}\left(-\frac{T}{2} h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu\right)$$

$$= \frac{\partial}{\partial \dot{X}_\mu}\left(-\frac{T}{2}\left(-\dot{X}^2 + X'^2\right)\right) = T\dot{X}^\mu.$$

and the canonical Hamiltonian

$$H_{can} = \int_0^\pi d\sigma' \left(\dot{X}_\mu \Pi^\mu - \mathcal{L}_P\right) = \int_0^\pi d\sigma' \left(T\dot{X}^2 + \frac{T}{2}\left(-\dot{X}^2 + X'^2\right)\right)$$

$$= \frac{T}{2}\int_0^\pi d\sigma' \left(\dot{X}^2 + X'^2\right) = T\int_0^\pi d\sigma_1 \left((\partial_+ X)^2 + (\partial_- X)^2\right).$$

$T_{++} + T_{--} = T_{00} = \frac{1}{2}(\dot{X}^2 + X'^2) = T_{11} =$

is the integrated version of the constraint. $T_{00} = T_{11} = 0$

world-sheet currents

we would like to find what conserved quantities we can associate with the string.

conserved quantities do not change in time and are useful for physical measurements and to characterize physical systems.

In physics there is a relation between symmetries and conserved quantities.

e.g.: translation $\Rightarrow$ linear momentum

LST. 74 / 23/10/09.1 / Abingdon / FRiday

Rotation → angular momentum.

we want to describe these statements in mathematical form.

A conserved current satisfies the equation:

$\partial_\alpha j^\alpha = 0$

e.g. in electromagnetism $j^\alpha = (c\rho, \vec{J})$.

(*) $\partial_\alpha j^\alpha = \partial_0 j^0 + \partial_i j^i = \frac{\partial j^0}{\partial x^0} + \vec{\nabla} \cdot \vec{j} = 0$.

The rate of change of the charge density is equal to the flux of the charge current.

equation (*) implies that electric charge is conserved.

Total charge: $Q(t) = \int_V \rho(t, \vec{x})\, d^3x = \int_V \frac{j^0(t,\vec{x})}{c}\, d^3x$.

$$\frac{dQ(t)}{dt} = \int_V \frac{\partial j^0}{\partial x^0}\, d^3x =$$

$$= \int_V -\nabla \cdot \vec{J}\, d^3x = \int_S -\vec{j} \cdot d\vec{a}$$

where S denotes the boundary of V.

⇒ charge can only change if there is a flux of current across the boundary, which bounds the volume

if $V \to \infty$ and $j_\infty = 0 \Rightarrow \frac{dQ}{dt} = 0 \to$ charge is conserved

LST.75 | 23/10/09.2 | Abingdon | FRiday |

Conserved charges from Lagrangian symmetries

consider the Lagrangian $L(q(t), \dot{q}(t), t)$

and variation of $q(t) \to q(t) + \delta(q(t)) = q(t) + \epsilon h(q(t), t)$

$$\dot{q}(t) \to \dot{q}(t) + \frac{d}{dt}\delta(q(t))$$

if the Lagrangian is invariant under the variation → symmetry

→ A conserved charge associated with the symmetry:

$$\epsilon Q \equiv \frac{\partial L}{\partial \dot{q}} \delta q \quad \text{and} \quad \frac{dQ}{dt} = 0.$$

Proof | $S = \int dt\, L \to \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0.$

since L is invariant under the variations.

$$L(q + \delta q, \dot{q} + \frac{d}{dt}\delta q, t) = L + \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}\delta q + \cdots = L$$

$$\Rightarrow \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}\delta q = 0.$$

now take | $\epsilon \frac{dQ}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\delta q + \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}(\delta q) =$

$$= \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}\delta q = 0 \Rightarrow \frac{dQ}{dt} = 0.$$

$$Q(t) = \frac{\partial L}{\partial \dot{q}} h(q(t), t) \quad \text{is conserved.}$$

LST, 76 / 23/10/09.3 /

For example: $h=1 \Rightarrow Q = \frac{\partial L}{\partial \dot{q}} = P \Rightarrow \frac{dP}{dt} = 0$.

momentum corresponds to invariance under translations.

The case of Lagrange densities:

$$S = \int d\eta^0 \dots d\eta^h \, \mathcal{L}(\phi^a, \partial_\alpha \phi^a)$$

$h \to$ number of dimensions.

$\phi^a \to$ fields $\equiv \phi^a(\eta)$ and $\partial_\alpha \phi^a = \frac{\partial \phi^a}{\partial \eta^\alpha}$

take: $\phi^a(\eta) \to \phi^a(\eta) + \delta\phi^a(\eta)$.

with $\delta \phi^a = \epsilon^i h_i^a(\phi)$. (summation over i).

statement: if $\mathcal{L}$ is invariant under the variation then.

$$\epsilon^i j_i^\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)} \delta \phi^a$$

are conserved currents and $\partial_\alpha j_i^\alpha = 0$.

j_i^α : i labels the various currents.

α labels the components of the currents.

$[h^i, \eta^\alpha, \phi^a]$

associated conserved charge: $Q_i = \int d\eta^1 \dots d\eta^h \, j_i^0$

and $\frac{dQ}{d\eta^0} = 0$ (proof similar the the earlier one).

$\hookrightarrow$ taking $V_h \to \infty$ and $j_\infty \to 0$.

LST. 77 | 23/10/09.4 | Abingdon | Friday |

PROOF | E.L. equation for fields: $\partial_\alpha\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\alpha \phi^a\right)}\right) - \frac{\partial \mathcal{L}}{\partial \phi^a} = 0.$

$\phi^a \to \phi^a + \delta\phi^a \Rightarrow \mathcal{L}(\phi^a + \delta\phi^a, \partial_\alpha \phi^a + \partial_\alpha \delta\phi^a) = \mathcal{L}(\phi^a, \partial_\alpha \phi^a)$

$+ \frac{\partial \mathcal{L}}{\partial_\alpha \phi^a}\delta\phi^a + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)}\partial_\alpha\delta\phi^a + \dots = \mathcal{L}(\phi^a, \partial_\alpha \phi^a).$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial_\alpha \phi^a}\delta\phi^a + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)}\partial_\alpha\delta\phi^a = 0. \qquad (*)$$

$$\Rightarrow \epsilon^i \partial_\alpha j_i^{\ \alpha} = \partial_\alpha\left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)}\delta\phi^a\right) =$$

$$= \partial_\alpha\left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)}\right)\delta\phi^a + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)}\partial_\alpha\delta\phi^a =$$

$$= \frac{\partial \mathcal{L}}{\partial \phi^a}\delta\phi^a + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)}\partial_\alpha\delta\phi^a = 0 \quad | \text{by}(*).$$

- Conserved currents on the world-sheet

in the case of the string: $\alpha = 0, 1.$
$a = \mu. \quad \phi^a = X^\mu.$

$S = \int d\eta^0 d\eta^1 \mathcal{L}(\partial_0 X^\mu, \partial_1 X^\mu) \qquad (\eta^0, \eta^1) = (\tau, \sigma).$

The string action is invariant under Poincare trans:

$X^\mu \to \Lambda^\mu{}_\nu X^\nu + a^\mu.$

Translations | $\delta X^\mu(\tau, \sigma) = \epsilon^\mu. \qquad \epsilon^\mu = \text{constant}.$

LST. 78 | 23/10/09.5 | Abingdon | Friday

→ Conserved currents | $\epsilon^\mu j_\mu^\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\mu)} \delta X^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\mu)} \epsilon^\mu$

$\Rightarrow j_\mu^\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\mu)} \longrightarrow (j_\mu^0, j_\mu^1) = \left(\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu}, \frac{\partial \mathcal{L}}{\partial X'^\mu}\right),$

identify $j_\mu^\alpha = \mathcal{P}_\mu^\alpha \rightarrow (j_\mu^0, j_\mu^1) = (\mathcal{P}_\mu^\tau, \mathcal{P}_\mu^\sigma).$

The current conservation equation coincides with the equation of motion for the string.

$\partial_\alpha j_\mu^\alpha = \partial_\alpha \mathcal{P}_\mu^\alpha = \frac{\partial}{\partial\tau}\mathcal{P}_\mu^\tau + \frac{\partial}{\partial\sigma}\mathcal{P}_\mu^\sigma = 0.$

The conserved charge follows from $Q_i = \int dy^1 \cdots dy^d \, j_i^0$.

$\Rightarrow p_\mu(\tau) = \int_0^\pi \mathcal{P}_\mu^\tau(\tau,\sigma)\, d\sigma \rightarrow$ space-time momentum carried by the string.

→ $\mathcal{P}_\mu^\tau(\tau,\sigma) \rightarrow \sigma$ density of the space-time momentum carried by string

check conservation | $\frac{d p_\mu}{d\tau} = \int_0^\pi \frac{\partial \mathcal{P}_\mu^\tau}{\partial\tau} d\sigma = -\int_0^\pi \frac{\partial \mathcal{P}_\mu^\sigma}{\partial\sigma} d\sigma = -\mathcal{P}_\mu^\sigma \Big|_0^\pi$

For closed string $\mathcal{P}_\mu^\sigma(\pi) = \mathcal{P}_\mu^\sigma(0)$ by periodicity.

For open string with Neumann B.C. $\frac{\partial \mathcal{L}}{\partial X'}\Big|_0 = \frac{\partial \mathcal{L}}{\partial X'}\Big|_\pi = 0$

$\Rightarrow$ in both cases: $\frac{d}{d\tau} p_\mu(\tau) = 0$

For Dirichlet B.C. → no translation invariance
→ momentum not conserved

LST. 79 | 23/10/09.6 | Abingdon | Friday |

Lorentz symmetry

The action is invariant under $X^\mu \to \Lambda^\mu{}_\nu X^\nu$

Q.M. - may be spoiled $\to$ constraints.

infinitesimal transformations $X^\mu \longrightarrow X^\mu + \delta X^\mu$

with $\delta X^\mu = \epsilon^{\mu\nu} X_\nu$ $\quad \epsilon^{\mu\nu} \to$ infintesimal constants.

Lorentz invariance requires: $\eta_{\mu\nu} X^\mu X^\nu = \eta_{\mu\nu} (X^\mu + \epsilon^{\mu\alpha} X_\alpha)(X^\nu + \epsilon^{\nu\beta} X_\beta)$

$$= \eta_{\mu\nu} X^\mu X^\nu + \eta_{\mu\nu} \epsilon^{\mu\alpha} X_\alpha X^\nu + \eta_{\mu\nu} X^\mu \epsilon^{\nu\beta} X_\beta + O(\epsilon$$

$$\Rightarrow \quad \epsilon^{\mu\alpha} X_\alpha X_\mu + \epsilon^{\nu\beta} X_\nu X_\beta = \epsilon^{\mu\nu} X_\nu X_\mu + \epsilon^{\nu\mu} X_\nu X_\mu$$

$$= (\epsilon^{\mu\nu} + \epsilon^{\nu\mu}) X_\nu X_\mu = 0 \Rightarrow \epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$$

$\to$ ϵ is antisymmetric

Recall $\epsilon^i j_i{}^\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)} \delta\phi^a \Rightarrow \epsilon^{\mu\nu} j_{\mu\nu}{}^\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\mu)} \delta X^\mu =$

$$= P_\mu{}^\alpha \epsilon^{\mu\nu} X_\nu$$

where $j_{\mu\nu}$ is antisymmetric under exchange of $\mu \leftrightarrow \nu$.

$\to$ $\epsilon^{\mu\nu} j_{\mu\nu}{}^\alpha = -\frac{1}{2} \epsilon^{\mu\nu} (X_\mu P_\nu{}^\alpha - X_\nu P_\mu{}^\alpha)$.

$\to$ up to normalization $M_{\mu\nu}{}^\alpha = X_\mu P_\nu{}^\alpha - X_\nu P_\mu{}^\alpha$

with $M_{\mu\nu}{}^\alpha = -M_{\nu\mu}{}^\alpha$

$\longrightarrow$ conserved currents.

LST, 8U | 25/10/09. 1 | Abingdon | sunday |

The equation of current conservation.

$$\frac{\partial \mathcal{M}^{\tau}}{\partial \tau} + \frac{\partial \mathcal{M}^{\sigma}}{\partial \sigma} = 0.$$

$\rightarrow$ charge $M_{\mu\nu} = \int \mathcal{M}^{\tau}_{\mu\nu}(\tau, \sigma)\, d\sigma = \int_0^{\pi} (X_{\mu} P_{\nu}^{\tau} - X_{\nu} P^{\tau}_{\mu})\, d\sigma$

on lines of constant τ.

six conserved charges in 4-dimensions:

$M_{0i} \rightarrow$ boosts $\quad L_i = \frac{1}{2} \epsilon_{ijk} M_{jk} \Rightarrow$ angular momentum

Fourier expansion

Recall: wave eq.: $\partial_{\alpha} \partial^{\alpha} X^{\mu} = \frac{\partial}{\partial \tau^2} X^{\mu} - \frac{\partial}{\partial \sigma^2} X^{\mu} = 0.$

or using the light-cone variables: $\sigma^{\pm} = \tau \pm \sigma$

$$\partial_{+} \partial_{-} X^{\mu} = 0.$$

The most general solution:

$$X^{\mu}(\sigma, \tau) = X^{\mu}_{L}(\sigma^{+}) + X^{\mu}_{R}(\sigma^{-}).$$

The solutions depend on the boundary conditions.

For closed string: $\quad X^{\mu}(\sigma, \tau) = X^{\mu}(\sigma + \pi, \tau).$

The most general solution for closed string B. C.

$$X_L^{\mu} = \frac{1}{2} X^{\mu} + \frac{1}{2} l_s^2 P^{\mu}(\tau + \sigma) + \frac{i}{2} l_s \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n^{\mu} e^{-i2n(\tau+\sigma)}$$

$$X_R^{\mu} = \frac{1}{2} X^{\mu} + \frac{1}{2} l_s P^{\mu}(\tau - \sigma) + \frac{i}{2} l_s \sum_{n \neq 0} \frac{1}{n} \alpha_n^{\mu} e^{-i2n(\tau-\sigma)}$$

LST.21 | 25/10/09.2 | Abingdon | sunday |

$$X^\mu = X_L^\mu + X_R^\mu = X^\mu_{com.} + l_s^2 p^\mu \tau + \frac{i}{2} l_s \sum_{n\neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-i2n(\tau+\sigma)} + \frac{i}{2} l_s \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-i2n(\tau-\sigma)}$$

where:

x^μ : centre of mass position.

p^μ : total string momentum.

The exponential terms represent the string excitation

l_s : string length. related to string tension. T

and the Regge slope parameter $(J/\hbar = \alpha' E^2)$.

with. $T = \frac{1}{2\pi\alpha'}$ $\frac{1}{2} l_s^2 = \alpha'$.

$\rightarrow l_s^2 = \frac{1}{\pi T}$

X_R^μ, X_L^μ are real functions $\Rightarrow$ x^μ, p^μ are real.

$$(\alpha_n^\mu)^* = \alpha_{-n} \qquad (\tilde{\alpha}_n^\mu)^* = \tilde{\alpha}^\mu_{-n}$$

α_n, α_{-n} are positive and negative fourier modes.

Q. M. : $\alpha_n, \alpha_{-n} \rightarrow a, a^+$ with $[a, a^+] = 1$

The total momentum of the string:

$$P^\mu = T\int_0^\pi d\sigma\, \dot{X}^\mu = T \frac{1}{\pi T} p^\mu \sigma \Big|_0^\pi = p^\mu$$

The centre of mass of the string moves along a straight line.

check | ? $x^\mu_{cms} = \frac{1}{\pi}\int_0^\pi d\sigma\, X^\mu(\sigma,\tau) = x^\mu + p^\mu \tau^0 = x^\mu(0) + \frac{dx^\mu}{d\tau}(0)\,\tau.$

$\longrightarrow$ String: relativistic particle plus L- and R-moving harmonic oscillations

LST.82 | Sunday | 25/10/09.3 | Abingdon |

Imposing the constraints

Recall: The constraints: $T_{01} = \dot{X} \cdot X' = 0$

$$T_{00} = T_{11} = \frac{1}{\alpha}(\dot{X}^2 + X'^2) = 0.$$

in light-cone coordinates: $T_{++} = (\partial_+ X)^2 = 0$

$$T_{--} = (\partial_- X)^2 = 0$$

give constraints on the momenta p^μ and the fourier modes α_n^μ, $\tilde{\alpha}_n^\mu$.

e.g. $\partial_- X^\mu = \partial_- X_R^\mu = \frac{1}{\alpha} l_s p^\mu + l_s \sum_{m \neq 0} \alpha_m^\mu e^{-2im(\tau - \sigma)}$

$$= l_s \sum_{m=-\infty}^{\infty} \alpha_m^\mu e^{-2im(\tau-\sigma)}$$

$\alpha_0^\mu = \frac{1}{2} l_s p^\mu$

$$(\partial_- X)^2 = l_s^2 \sum_{m,p} \alpha_m \cdot \alpha_p e^{-2i(m+p)(\tau-\sigma)}$$

$$= l_s^2 \sum_{m,n} \alpha_m \cdot \alpha_{n-m} e^{-2in(\tau-\sigma)}$$

$$= 2 l_s^2 \sum_n L_n e^{-2in(\tau-\sigma)} = 0.$$

where the sum over oscillation modes is defined by.

$$L_n = \frac{1}{2} \sum_m \alpha_{n-m} \cdot \alpha_m$$

and similarly for the left-movers.

$$\tilde{L}_n = \frac{1}{2} \sum_m \tilde{\alpha}_{n-m} \cdot \tilde{\alpha}_m. \quad \text{with } \tilde{\alpha}_0^\mu = \frac{1}{\alpha} l_s p^\mu.$$

The constraints $T_{++} = T_{--} = 0$

translate to $L_n = \tilde{L}_n = 0 \qquad n \in \mathbb{Z}$

LST.P3 | 25/10/09.4 | Abingdon | Sunday |

The L_m and $\tilde{L}_m$ are fourier modes of the constraints.

e.g. $L_m = \frac{T}{2}\int_0^{\pi} e^{-2im\sigma} T_{--} d\sigma$ (evaluated at $\tau = 0$).

and similarly for $\tilde{L}_m$

$L_m, \tilde{L}_m$ are the fourier modes of the world-sheet

energy-momentum tensor and are called the Virasoro operators

Hamiltonian | world-sheet time evolution is generated by the Hamiltonian

$$H = \int_0^{\pi} d\sigma(\dot{X}_\mu P_\tau^\mu - \mathcal{L}) = \frac{T}{2}\int_0^{\pi} d\sigma(\dot{X}^2 + X'^2)$$

Where $P_\tau^\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}_\mu} = T\dot{X}^\mu$.

Inserting the mode expansion for the closed string

we have | $$H = \sum_{n=-\infty}^{+\infty}(\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_{-n}\cdot\alpha_n) = 2(L_0 + \tilde{L}_0) \qquad (H,N)$$

we can use the constraint

$$L_0 = \tilde{L}_0 = 0.$$

To derive an expression for the mass of a string.

The relativistic mass shell condition

$$p_\mu p^\mu = -M^2.$$

where $P_\mu = T\int_0^{\pi} d\sigma\, X^\mu(\sigma)$ is the total momentum of the string.

LST. RY | 25/10/09. 26 | Abingdon | Sunday |

For closed string.

$$L_0+\tilde{L}_0 = \frac{1}{2}\sum\left(\alpha_{-m}\cdot\alpha_m + \tilde{\alpha}_{-m}\cdot\tilde{\alpha}_m\right) =$$

$$= \frac{1}{2}\alpha_0^{\mu}\alpha_{0\mu} + \frac{1}{2}\tilde{\alpha}_0^{\mu}\tilde{\alpha}_{0\mu} + \frac{1}{2}\sum_{m\neq 0}\left(\alpha_{-m}\alpha_m + \tilde{\alpha}_{-m}\cdot\tilde{\alpha}_m\right)$$

$$= \frac{1}{2}\left(\frac{1}{4}l_s^2 P^{\mu}P_{\mu}\right)\cdot 2 + \frac{1}{2}\sum_{m\neq 0}\left(\alpha_{-m}\alpha_m + \tilde{\alpha}_{-m}\cdot\alpha_m\right) = 0.$$

$$\Rightarrow M^2 = \frac{2}{l_s^2}\sum_{m\neq 0}\left(\alpha_{-m}\cdot\alpha_m + \tilde{\alpha}_{-m}\cdot\alpha_m\right)$$

$$= \frac{2}{\alpha'}\sum_{n=1}^{\infty}\left(\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_{-n}\cdot\alpha_n\right)$$

→ mass shell condition for the string.

→ Determine the mass of a given string state.

defining The total occupation number.

$$N = \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n \qquad \tilde{N} = \sum_{n=1}^{\infty}\tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n = \sum_{n=1} n\tilde{a}_n^{\dagger}\tilde{a}_n.$$

we have: $M^2 = \frac{2}{\alpha'}(N+\tilde{N})$

note that the constraint. $L_0 = \tilde{L}_0$

$$\Rightarrow N = \tilde{N}.$$

which is called level matching because it implies that left and Right-moving modes contribute equally to the mass.

LST.85 | 30/10/09.1 | Liverpool | Friday | $T=\frac{1}{2\pi\alpha'}$ $\alpha'=\frac{l_s^2}{2} \Rightarrow T=\frac{1}{\pi l_s^2}$

open string mode expansion.

Neumann B.C.: $\left.\frac{\partial X^\mu}{\partial\sigma}\right|_{\sigma=0} = \left.\frac{\partial X^\mu}{\partial\sigma}\right|_{\sigma=\pi} = 0$

general solution: $X^\mu = x^\mu + l_s^2 p^\mu \tau + i l_s \sum_{m\neq 0} \frac{1}{m}\alpha_m^\mu e^{-im\tau}\cos(m\sigma)$

$\boxed{\alpha_0 = l_s p^\mu}$ This can still be decomposed into left and right-moving parts $X = X_L(\sigma^+) + X_R(\sigma^-)$

$X^\mu_{L/R} = \frac{1}{2}x^\mu + \frac{1}{2}l_s^2 p^\mu_{L/R}\sigma^\pm + \frac{i}{2}l_s\sum_{m\neq 0}\frac{1}{m}\alpha^\mu_{m(L/R)}e^{-im\sigma^\pm}$

But the boundary conditions imply

$X^\mu = X_L^\mu + X_R^\mu = x^\mu + \frac{1}{2}l_s^2\left[(p_L^\mu + p_R^\mu)\tau + (p_L^\mu - p_R^\mu)\sigma\right] + \ldots$

$+ \frac{il_s}{2}\sum_{m\neq 0}\frac{1}{m}\alpha^\mu_{mL}\left(\cos m(\tau+\sigma) - i\sin m(\tau+\sigma)\right) + \frac{1}{m}\alpha^\mu_{mR}\left(\cos m(\tau-\sigma) - i\sin m(\tau-\sigma)\right)$

$\frac{\partial X}{\partial\sigma} = \frac{1}{2}(p_L^\mu - p_R^\mu) + \frac{i l_s}{2}\sum \alpha^\mu_{mL}\left(-\sin m(\tau+\sigma) - i\cos m(\tau+\sigma)\right) + \alpha^\mu_{mR}\left(\sin m(\tau-\sigma) + i\cos m(\tau-\sigma)\right)$

$= \frac{1}{2}(p_L^\mu - p_R^\mu) + \frac{il_s}{2}\sum\left(-\alpha^\mu_{mL} + \alpha^\mu_{mR}\right)\left(\sin m(\tau+\sigma) + i\cos m(\tau-\sigma)\right) \rightarrow \sigma = 0.$

hence $\left.\frac{\partial X}{\partial\sigma}\right|_{\sigma=0,\pi} = 0 \Rightarrow p_L = p_R = \frac{1}{2}p$ $\alpha_{mL} = \alpha_{mR}.$

$\Rightarrow$ Left & right moving waves combine into standing waves.

$\Rightarrow \frac{1}{2}$ the number of independent modes compared to closed string.

LST. 86 | 30/10/09.2 | Liverpool / Friday |

combining B.C. with constraint.

$\dot{X}^2 + X'^2 = 0 \Rightarrow \dot{X}_\mu X'^\mu \Big|_{\sigma=0,\pi} = 0$

$\Rightarrow$ The end point of the open string move at the speed of light.

since X_L and X_R are related by $\sigma \to -\sigma$.

They can be combined into a single periodic field with period 2π.

The fourier decomposition gives one set of modes.

$$L_m = 2T\int_0^\pi d\sigma \left(e^{im\sigma} T_{++} + e^{-im\sigma} T_{--}\right) =$$

$$= \frac{1}{2}\pi T \sum_n \alpha_{m-n}\alpha_n. \qquad \text{where } \alpha_0 = l_s p^\mu$$

The canonical Hamiltonian is $H = L_0 = 0$.

The mass shell condition: $M^2 = -p^2 = 2\pi T N$.

Dirichlet B.C.

Linear term in σ but no linear term in τ.

For the oscillators $\cos \to \sin$

$\Rightarrow \dot{X}\Big|_{\sigma=0,\pi} = 0 \Leftrightarrow$ ends don't move.

but $X'\Big|_{\sigma=0,\pi} \neq 0 \to$ momentum is exchanged with the D-brane.

LST.87 | 30/10/09.3 | Liverpool | Friday

The quantized relativistic string

To proceed to quantize the string we can proceed in two directions:

(a) covariant quantization:

Impose $[X^\mu(\sigma,\tau), \Pi_\nu(\sigma',\tau)] = i(\delta\sigma-\sigma') \delta^\mu{}_\nu$

$$[X^\mu, X^\nu] = [\Pi^\mu, \Pi^\nu] = 0$$

analog of QFT equal-time comm. relations.

Then impose constraints: $X'\cdot\dot{X} = \dot{X}^2 + X'^2 = 0$

which selects physical states.

(b) solve constraints to select classically allowed solutions. quantize the physical solutions → Light-cone-quantization

The two methods are equivalent and yield the same physical results.

we will follow the first and then ditch in favor of the second.

The procedure is similar to that taken in quantization of quantum field theories.

LST. 8 | 30/10/09. 4 | Liverpool | Friday |

Lightning review of quantization.

quantization of the simple harmonic oscillator in one dimension.

classically: $H = \frac{P^2}{2m} + \frac{1}{2} m \omega^2 X^2$ $\quad V(X) = \frac{1}{2} k X^2$

$\omega^2 = k/m$.

Q. M. $\hat{H}|n\rangle = \left\{ \frac{\hat{P}^2}{2m} + \frac{1}{2} m\omega^2 \hat{X}^2 \right\} |n\rangle = \epsilon_n |n\rangle$ $\to \hat{H}, \hat{P}, \hat{X}$: Q. operators.

ϵ_n : E. eingenvalues.

canonical variables: $[\hat{X}, \hat{P}] = i\hbar \hat{1}$

define: $a = \left(\frac{m\omega}{2\hbar}\right)^{1/2} \hat{X} + i \frac{1}{(2m\hbar\omega)^{1/2}} \hat{P}$.

$a^+ = \left(\frac{2m}{2\hbar}\right)^{1/2} \hat{X} - i \frac{1}{(2m\hbar\omega)^{1/2}} \hat{P}$.

Then.

$aa^+ = \left[\left(\frac{m\omega}{2\hbar}\right)^{1/2} \hat{X} + \frac{i}{(2m\hbar\omega)^{1/2}} \hat{P}\right]\left[\left(\frac{m\omega}{2\hbar}\right)^{1/2} - \frac{i}{(2m\hbar\omega)^{1/2}} \hat{P}\right]$

$= \left(\frac{m\omega}{2\hbar}\right) \hat{X}^2 + \frac{\hat{P}^2}{(2m\hbar\omega)} + i \frac{\hat{P}\hat{X}}{2\hbar} - i \frac{\hat{X}\hat{P}}{2\hbar}$

$= \left(\frac{m\omega}{2\hbar}\right) \hat{X}^2 + \frac{\hat{P}^2}{(2m\hbar\omega)} - \frac{i}{2\hbar} [\hat{X}, \hat{P}]$.

similarly $a^+a = \left(\frac{m\omega}{2\hbar}\right) \hat{X}^2 + \frac{\hat{P}^2}{(2m\hbar\omega)} + \frac{i}{2\hbar} [\hat{X}, \hat{P}]$.

$\Rightarrow (aa^+ - a^+a) = -\frac{i}{2\hbar}[\hat{X}, \hat{P}] - \frac{i}{2\hbar}[\hat{X}, \hat{P}] = \frac{-i}{2\hbar} 2i\hbar = 1$

LST. 89 | 1/11/09. | Liverpool | sunday |

hence $[a, a^+] = \hat{1}$.

and $\hbar w a^+ a = \frac{\hat{P}^2}{2m} + \frac{1}{2} m w^2 \hat{x}^2 - \frac{1}{2}\hbar w = \hat{H} - \frac{1}{2}\hbar w$

$\Rightarrow \hat{H} = \hbar w \left(a^+ a + \frac{1}{2} \right) = \hbar w (\hat{N} + \frac{1}{2})$.

we need to find the eigenvalues of $\hat{N}|n\rangle = N_n |n\rangle$.

Then $\hat{H}|n\rangle = \epsilon_n |n\rangle = \hbar w (N_n + \frac{1}{2})$.

take $a(\hat{N}|n\rangle) = a a^+ a |n\rangle = (a^+ a + 1) a |n\rangle$

$= N_n (a|n\rangle)$.

$\Rightarrow a^+ a (a|n\rangle) = (N_n - 1)(a|n\rangle)$.

$\Rightarrow$ The state $a|n\rangle$ has eigen value $N_n - 1$.

$\Rightarrow$ a is a lowering operator.

(*) $a|n\rangle = C_n^- |n-1\rangle$ $\quad C_n^- \to$ constant.

similarly: $a^+(a^+ a|n\rangle) = a^+(a a^+ - 1)|n\rangle = (a^+ a - 1) a^+|n\rangle$

$= N_n (a^+|n\rangle)$.

$\Rightarrow a^+ a (a^+|n\rangle) = (N_n + 1)(a^+|n\rangle)$.

$\Rightarrow$ the state $a^+|n\rangle$ has eigenvalue $N_n + 1$.

$\Rightarrow$ a^+ is a raising operator.

(**) $a^+|n\rangle = C_n^+ |n+1\rangle$ $\quad C_n^+ \to$ constant

LST.90 | 1/1/09.2 | Liverpool | sunday

multiplying both relations (*) and (**) by their conjugates we get.

$\langle n|a^+a|n\rangle = |C_n^-|^2 \langle n-1|n-1\rangle \Rightarrow \langle n|\hat{N}|n\rangle = N_n = |C_n^-|^2$

and $\langle n|a\,a^+|n\rangle = |C_n^+|^2 \langle n+1|n+1\rangle \Rightarrow \langle n|1+\hat{N}|n\rangle = (1+N_n) = |C_n^+|^2$

From $N_n = |C_n^-|^2 \Rightarrow N_n \geq 0. \Rightarrow$ must exist $N_{n_{min}}$.

$\Rightarrow a|n_{min}\rangle = C_{n_{min}}|n-1\rangle = 0$

Define Vacuum: $\to |0\rangle$ state of lowest energy. $a|0\rangle = 0$.

$\Rightarrow N_n = n \Rightarrow \epsilon_n = \hbar\omega(n+\tfrac{1}{2}) \qquad n = 0,1,2\cdots$

$\Rightarrow C_n^- = \sqrt{n}$ and $C_n^+ = \sqrt{n+1}$

$|n-1\rangle = \frac{1}{\sqrt{n}}\, a|n\rangle \;;\; |n+1\rangle = \frac{1}{\sqrt{n+1}}\, a^+|n\rangle$

$|1\rangle = \frac{1}{\sqrt{1}}\, a^+|0\rangle \;\cdots\cdots\!> |n\rangle = \frac{1}{(n!)^{1/2}}\,(a^+)^n|0\rangle$

Quantization of fields

consider the Klein-Gordon Field:

K.G. EOM: $(\partial_\mu\partial^\mu + m^2)\,\Phi(x) = 0.$

Lagrangian density: $\mathcal{L} = \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}m^2\Phi^2$.

$= \frac{1}{2}\dot{\Phi}^2 - \frac{1}{2}(\vec{\nabla}\Phi)^2 - \frac{1}{2}m^2\Phi^2$ $\quad \eta = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

$\Pi = \frac{\partial\mathcal{L}}{\partial\dot{\Phi}} = \dot{\Phi},$

LST, 91 | 1/11/09.3 | Liverpool | Sunday |

Equal time commutation relations.

$$[\phi(\vec{x},t), \pi(\vec{x}',t)] = i\hbar\, \delta^3(\vec{x}-\vec{x}')$$

$$[\phi(\vec{x},t), \phi(\vec{x}',t)] = [\pi(\vec{x},t), \pi(\vec{x}',t)] = 0$$

Expand ϕ in fourier modes of momentum eigenstates.

$$\phi(\vec{x},t) = \frac{1}{(2\pi)^3}\int \frac{d^3\vec{p}}{2p^0}\left\{ a(\vec{p})e^{-ip\cdot x} + a^+(\vec{p})e^{+ip\cdot x}\right\}$$

Imposing equal time commutation relations implies,

$$[a(\vec{p}), a^+(\vec{p}')] = 2p^0\, \delta^3(\vec{p}-\vec{p}')(2\pi)^3$$

$$[a(\vec{p}), a(\vec{p}')] = 0$$

$$[a^+(\vec{p}), a^+(\vec{p}')] = 0$$

where $a(\vec{p})|0\rangle = 0 \quad \forall(\vec{p}) \quad \langle 0|0\rangle = 1$.

single particle state $|\vec{p}\rangle = a^+(\vec{p})|0\rangle$

two particle state $|\vec{p}_1, \vec{p}_2\rangle = a^+(\vec{p}_1)\, a^+(\vec{p}_2)|0\rangle$

etc.

Back to strings (with $c = \hbar = 1$)

in Q.M. $[x^i, p^j] = i\delta^{ij}$ or $[X^\mu, p^\nu] = i\eta^{\mu\nu}$.

in QFT: $[\phi(x), \phi(y)]_{x^0=y^0} = i\delta^3(\vec{x}-\vec{y})$

similarly | Free relativistic string: $[X^\mu(\tau,\sigma), \Pi^\nu(\tau',\sigma')]_{\tau=\tau'} = i\eta^{\mu\nu}\delta(\sigma-\sigma')$

LST.92 | 1/1/94 | Liverpool | Sunday |

and

$$[X^\mu(\tau,\sigma), X^\nu(\tau',\sigma')]_{\tau=\tau'} = [\Pi^\mu(\tau,\sigma), \Pi^\nu(\tau',\sigma')]_{\tau=\tau'} = 0.$$

where $\delta(\sigma) = \delta(\sigma+\pi)$ is the periodic δ-function.

and $$\delta(\sigma) = \frac{1}{\pi}\sum_{k=-\infty}^{\infty} e^{-2\pi i k\sigma/\pi} = \frac{1}{\pi}\sum_{k=-\infty}^{\infty} e^{-2ik\sigma}.$$

conformal gauge $\longrightarrow \Box X^\mu = 0 \longrightarrow$ solutions:

closed: $$X^\mu(\tau,\sigma) = x^\mu + l_s^2 p^\mu \tau + \frac{i}{2} l_s \sum_{n\neq 0} \frac{1}{n}\tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)} + \frac{i}{2} l_s \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-2in(\tau-\sigma)}$$

open: $$X^\mu(\tau,\sigma) = x^\mu + l_s^2 p^\mu \tau + i l_s \sum_{m\neq 0} \alpha_m^\mu e^{-im\tau}\cos(m\sigma).$$

canonical momentum | $$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 X_\mu)} = T\dot{X}^\mu =$$

Imposing the quantization conditions on $[X^\mu, \Pi^\nu] = i\eta^{\mu\nu}\delta(\sigma-\sigma')$ etc...

imposes |
$$[x^\mu, p^\nu] = i\eta^{\mu\nu}$$
$$[\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+n,0}\,\eta^{\mu\nu}$$
$$[\alpha_m^\mu, \tilde{\alpha}_n^\nu] = 0.$$
$$[\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\delta_{m+n,0}\,\eta^{\mu\nu}$$

Hermiticity of X^μ implies:

$$(x^\mu)^\dagger = x^\mu \qquad (p^\mu)^\dagger = p^\mu \qquad (\alpha_m^\mu)^\dagger = \alpha_{-m}^\mu.$$

The commutation relations are similar to harmonic oscillator

LST.93 | 8/11/09.1 | Abingdon | Sunday |

Defining

$$a_m^{\mu} = \frac{1}{\sqrt{m}} \alpha_m^{\mu} \qquad a_m^{\mu +} = \frac{1}{\sqrt{m}} \alpha_{-m}^{\mu} \quad \text{for } m>0.$$

$$\rightarrow [a_m^{\mu}, a_n^{\nu +}] = [\tilde{a}_m^{\mu}, \tilde{a}_n^{\nu +}] = \eta^{\mu\nu} \delta_{m,n}$$

which generalize the commutator $[a, a^{+}] = 1$ of the SHO.

Note! $[a_n^0, a_m^{0+}] = -1$ → the commutator of time components have a negative sign.

⇒ negative norm states - ghosts.

we construct the space of states by acting with creation operators. → Fock space + constraints → Hilbert space (physical states).

ground state: $|0\rangle$ such that $\alpha_m^{\mu} |0\rangle = 0$ for $m>0$

$$p^{\mu} |0\rangle = 0.$$

momentum eigenstates are defined by:

$$p^{\mu} |k\rangle = k^{\mu} |k\rangle$$

where $|k\rangle = e^{ik.x} |0\rangle$, $k \cdot x = k_{\mu} x^{\mu}$.

single particle string states are obtained by acting with creation operators:

$\alpha_{-m}^{\mu} |0\rangle \rightarrow N_m^{\mu} = 1$ - m^{th} mode in the μ-direction.

a general excited state: $\alpha_{-m_1}^{\mu_1} \alpha_{-m_2}^{\mu_2} \cdots |0\rangle \rightarrow \{N_m^{\mu}\}$

A basis for the Fock space is obtained by taking linear combinations of states of the form:

$$\alpha_{-m_1}^{\mu_1} \alpha_{-m_2}^{\mu_2} \cdots |k\rangle.$$

which carries momentum k_μ and Excitation numbers

$$\{ N_m^\mu \mid \mu = 0, \ldots, D-1; \; m > 0 \}.$$

The ground state decomposes into:

$$|0\rangle = |0\rangle_{osc} \otimes |0\rangle_{mom}$$

with ${}_{osc}\langle 0|0\rangle_{osc} = 1$.

we define a scalar product in this space

with $\left(\alpha_{-m}^\mu |0\rangle, \alpha_{-n}^\nu |0\rangle\right) = {}_{osc}\langle 0|\alpha_m^\mu \alpha_{-n}^\nu |0\rangle_{osc} =$

$= {}_{osc}\langle 0| [\alpha_m^\mu, \alpha_{-n}^\nu] |0\rangle_{osc} = m\eta^{\mu\nu} \delta_{m+n,0}$

→ There are states with negative norm for $\mu=\nu=0$.

→ still need to implement the constraints,

scalar product between momentum eigenstates

$\langle k|k'\rangle = \delta^D(k-k')$ → undefined for $k=k'$.

→ unrenormalizable. → form momentum wave-packets

$$|\phi\rangle = \int d^D k \; \phi(k)|k\rangle$$

with $\langle \phi|\phi\rangle = \int d^D k \; \overline{\phi}(k)\phi(k) < \infty$

LST.95 | 8/11/09.3 | Abingdon | Sunday

implementing the constraints

we need to impose the constraints $L_m = 0 = \tilde{L}_m$

to extract the physical states. $L_m = \frac{1}{\alpha} \sum_{n=-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n$

quantum mechanically we require that:

$$L_n |Phys\rangle = \tilde{L}_n |Phys\rangle = 0 \qquad \text{For } n > 0$$

Q.M. we have to require that all creation - annihilation operators are normal ordered

$$:\alpha_m^\mu \alpha_n^\nu: = \begin{cases} \alpha_m^\mu \alpha_n^\nu & \text{if } m<0 \quad n>0 \\ \alpha_n^\nu \alpha_m^\mu & \text{if } m>0 \quad n<0 \\ \text{any ordering if else} \end{cases}$$

From the commutation relation:

$$[\alpha_m^\mu, \alpha_n^\nu] = m\,\delta_{m+n,0}\,\eta^{\mu\nu} \qquad [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\,\delta_{m+n,0}\,\eta^{\mu\nu}$$

$\Rightarrow$ normal ordering has a non-trivial effect when $m = n$ $\mu = \nu$.

$\rightarrow$ Normal ordering ambiguity arises in L_m.

only when $m - n = -n$ i.e. For $m = 0$.

Normal ordered version of L_0.

$$L_0^{NO} = \frac{1}{2} : \sum_{n=-\infty}^{\infty} : \alpha_{-n} \cdot \alpha_n : = \frac{1}{2} \sum_{n=-\infty}^{-1} : \alpha_{-n} \cdot \alpha_n + \frac{1}{2} \alpha_0 \cdot \alpha_0 + \frac{1}{2} \sum_{n=1}^{\infty} : \alpha_{-n} \alpha_n$$

$$= \frac{1}{2} p^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n = \frac{1}{2} p^2 + N$$

LST.96 | 8/11/09.4 | Abingdon | Sunday

Compare with a another. ordered version of L_0.

$$L_0^{CO} = \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{-n}\cdot\alpha_n = \frac{1}{2}\sum_{n=-\infty}^{-1}\alpha_{-n}\alpha_n + \frac{1}{2}\alpha_0\cdot\alpha_0 + \frac{1}{2}\sum_{n=1}^{\infty}\alpha_{-n}\alpha_n$$

$$= \frac{1}{8}P^2 + \frac{1}{2}\sum_{n=1}^{\infty}(\alpha_n\cdot\alpha_{-n} + \alpha_{-n}\cdot\alpha_n)$$

$$= \frac{1}{8}P^2 + \frac{1}{2}\sum_{n=1}^{\infty}(\alpha_n^{\mu}\alpha_{-n\mu} + \alpha_{-n}^{\mu}\alpha_{n\mu}) = \frac{1}{8}P^2 + \frac{1}{2}\sum_{n=1}^{\infty}\eta_{\mu\nu}(\alpha_n^{\mu}\alpha_{-n}^{\nu} + \dots$$

Recall | $[\alpha_m^{\mu}, \alpha_n^{\nu}] = m\delta_{m+n,0}\,\eta^{\mu\nu} \rightarrow [\alpha_m^{\mu}, \alpha_{-m}^{\nu}] = m\,\eta^{\mu\nu}$

$$\alpha_m^{\mu}\alpha_{-m}^{\nu} - \alpha_{-m}^{\nu}\alpha_m^{\mu} = m\,\eta^{\mu\nu}$$

$$\Rightarrow = \frac{1}{8}P^2 + \sum_{n=1}(\alpha_{-n}\cdot\alpha_n + \frac{n\,\eta^{\mu\nu}}{2}\eta_{\mu\nu})$$

Hence | $$L_0^{CO} = L_0^{NO} + \frac{D}{2}\sum_{n=1} n$$

i.e. the two versions of L_0 differ by an infinite constant.

→ consequence of summing over an infinite # of harmonic oscillators

A similar problem arises in QFT, in which case we can ignore it, because

(a): Gravity is decoupled

(b): QFT is formulated in the infinite volume limit.

In string theory gravity is coupled, and we work with finite size strings. → vacuum energy cannot be ignored