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Classical Relativistic String

Point particle – Action – world-line. integral over ds.

String particle – Action – world-area (sheet) integral over dA.

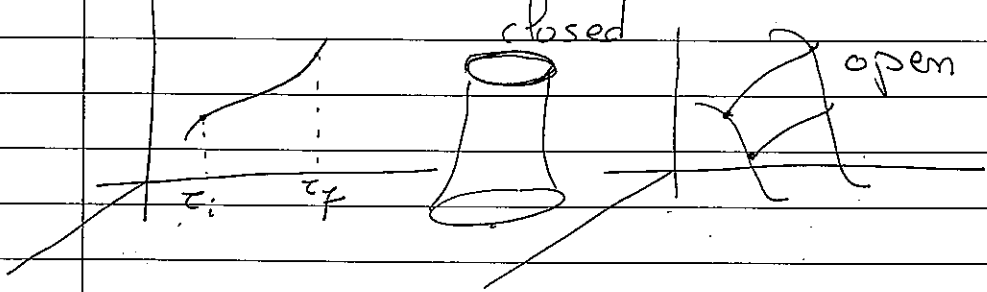
P. Particle: world-line – one parameter – τ – 'proper time'.

String: world-area – two parameters – $\tau, \sigma \to$ parameter space.

higher order membranes (branes) – $\tau, \sigma, \rho, \ldots$

Target space. P. Particle $X^\mu(\tau)$.

S. Particle $X^\mu(\tau, \sigma)$.



Area functional | In 3 spatial dimensions – 2 parameters η_1, η_2.

$$\vec{X}(\eta_1, \eta_2) = \left(X(\eta_1, \eta_2), \; y(\eta_1, \eta_2), \; z(\eta_1, \eta_2) \right).$$

What is the Domain of the parameters η_1, η_2?

We are parametrizing a finite area hence take $\eta_1, \eta_2 \in [0, \pi]$

The case of a propogating string will be different.

τ, σ: $-\infty \leq \tau \leq +\infty \quad 0 \leq \sigma \leq \pi$ ← in some units.

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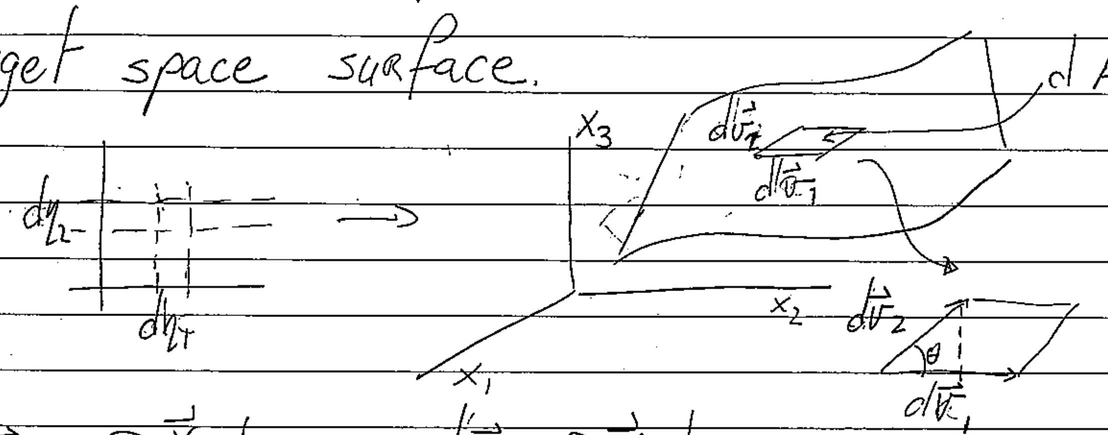
$\eta_1, \eta_2 \Rightarrow$ parameters

The physical space: $\vec{X}(\eta_1, \eta_2) \rightarrow$ target-space.

question | what is the area of a small element of the target space surface.



(*) $d\vec{v}_1 = \dfrac{\partial \vec{X}}{\partial \eta_1} d\eta_1 \qquad d\vec{v}_2 = \dfrac{\partial \vec{X}}{\partial \eta_2} d\eta_2.$

$$dA = |d\vec{v}_1| \cdot |d\vec{v}_2| \, |\sin\theta|$$

$$= |d\vec{v}_1| \, |d\vec{v}_2| \sqrt{1 - \cos^2\theta}$$

$$= \sqrt{|d\vec{v}_1|^2 |d\vec{v}_2|^2 - (|d\vec{v}_1||d\vec{v}_2|\cos\theta)^2}$$

$$= \left((d\vec{v}_1 \cdot d\vec{v}_1) \cdot (d\vec{v}_2 \cdot d\vec{v}_2) - (d\vec{v}_1 \cdot d\vec{v}_2)^2 \right)^{1/2}$$

using (*) $$= d\eta_1 d\eta_2 \left[\left(\frac{\partial \vec{X}}{\partial \eta_1} \cdot \frac{\partial \vec{X}}{\partial \eta_1} \right)\left(\frac{\partial \vec{X}}{\partial \eta_2} \cdot \frac{\partial \vec{X}}{\partial \eta_2} \right) - \left(\frac{\partial \vec{X}}{\partial \eta_1} \cdot \frac{\partial \vec{X}}{\partial \eta_2} \right)^2 \right]^{1/2}$$

The Area functional $\quad A = \int dA$

Reparameterization invariance of the area

The area should be independent of the parameterization

Reparameterization Variance - fundamental concept.

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take $\tilde{\eta}_1 = \tilde{\eta}_1(\eta_1, \eta_2)$ $\quad \tilde{\eta}^2 = \tilde{\eta}_2(\eta_1, \eta_2)$

$$d\eta_1 d\eta_2 = \left|\det\left(\frac{\partial \eta^i}{\partial \tilde{\eta}_j}\right)\right| d\tilde{\eta}_1 d\tilde{\eta}_2 = |\det M| \, d\tilde{\eta}_1 d\tilde{\eta}_2$$

M_{ij} is the matrix given by $M_{ij} = \dfrac{\partial \eta_i}{\partial \tilde{\eta}_j}$

similarly: $d\tilde{\eta}_1 d\tilde{\eta}_2 = \left|\det\left(\dfrac{\partial \tilde{\eta}^i}{\partial \eta_j}\right)\right| d\eta_1 d\eta_2 = |\det \tilde{M}| \, d\eta_1 d\eta_2.$

where $\tilde{M}_{ij} = \dfrac{\partial \tilde{\eta}^i}{\partial \eta^j}$

$$\Rightarrow |\det M| \, |\det \tilde{M}| = 1$$

consider the surface S described by the mapping function $\vec{X}(\eta_1, \eta^2)$.

Given $d\vec{X}$ - Vector tangent to the surface.

Then $ds^2 \equiv (ds)^2 = d\vec{X} \cdot d\vec{X}$. (no minus sign)

$$d\vec{X} = \frac{\partial \vec{X}}{\partial \eta_1} d\eta_1 + \frac{\partial \vec{X}}{\partial \eta_2} d\eta_2 = \frac{\partial \vec{X}}{\partial \eta^i} d\eta^i \quad \leftarrow \text{summ}$$

Then: $ds^2 = \left(\dfrac{\partial \vec{X}}{\partial \eta^i} d\eta^i\right) \cdot \dfrac{\partial \vec{X}}{\partial \eta^j} d\eta^j = \dfrac{\partial \vec{X}}{\partial \eta^i} \dfrac{\partial \vec{X}}{\partial \eta^j} d\eta^i d\eta^j.$

OR $ds^2 = g_{ij}(\eta^1, \eta^2) \, d\eta^i d\eta^j.$

where: $g_{ij}(\eta^1, \eta^2) = \dfrac{\partial \vec{X}}{\partial \eta^i} \dfrac{\partial \vec{X}}{\partial \eta^j} \rightarrow$ the induced metric on S.

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The determinant of g_{ij} is given by

$$g \equiv \det(g_{ij}) = \left(\frac{\partial \vec{X}}{\partial \eta_1} \cdot \frac{\partial \vec{X}}{\partial \eta_1}\right) \cdot \left(\frac{\partial \vec{X}}{\partial \eta_2} \cdot \frac{\partial X}{\partial \eta_2}\right) - \left(\frac{\partial \vec{X}}{\partial \eta_1} \cdot \frac{\partial \vec{X}}{\partial \eta_2}\right)^2$$

The area functional is then

$$A = \int d\eta^1 d\eta^2 \sqrt{g}$$

We can this form to study reparameterization ivariance of the area functional. in term of the transformation properties of g_{ij}.

<u>under reparameterization</u> $\eta_i, \eta_j \to \tilde{\eta}_{i,j}(\eta_i, \eta_j)$

equality $$g_{ij}(\eta) d\eta^i d\eta^j = \tilde{g}_{pq}(\tilde{\eta}) d\tilde{\eta}^p d\tilde{\eta}^q$$

$$= \tilde{g}_{pq}(\tilde{\eta}) \frac{\partial \tilde{\eta}^p}{\partial \eta^i} \frac{\partial \tilde{\eta}^q}{\partial \eta^j} d\eta^i d\eta^j.$$

$$\Rightarrow \quad g_{ij}(\eta) = \tilde{g}_{p,q}(\tilde{\eta}) \frac{\partial \tilde{\eta}^p}{\partial \eta^i} \frac{\partial \tilde{\eta}^q}{\partial \eta^j}$$

OR $$g_{ij}(\eta) = \tilde{g}_{p,q}(\tilde{\eta}) \tilde{M}_{pi} \tilde{M}_{qj}.$$

$$= (\tilde{M}^T)_{ip} \tilde{g}_{p,q}(\tilde{\eta}) \tilde{M}_{qj}.$$

taking the determinant <u>on</u> both sides,

$$\Rightarrow \quad g = (\det \tilde{M}^T) \tilde{g} (\det \tilde{M}) = \tilde{g} (\det \tilde{M})^2$$

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- OR $\sqrt{g} = \sqrt{\tilde{g}}\,|\det \tilde{M}|$

Recalling the expressions:

$$d\eta^1 d\eta^2 = |\det M|\, d\tilde{\eta}_1\, d\tilde{\eta}_2$$

and $|\det M||\det \tilde{M}| = 1$

} that we saw before.

$$\Rightarrow \int d\eta^1 d\eta^2 \sqrt{g} = \int d\tilde{\eta}_1 d\tilde{\eta}_2 |\det M|\cdot|\det \tilde{M}| \sqrt{\tilde{g}} = \int d\tilde{\eta}_1 d\tilde{\eta}_2 \sqrt{\tilde{g}}$$

- → Reparameterization invariance -

Area functional for space-time surfaces

Parmeters of the world-sheet of the propagating string: τ, σ.

τ - "Proper-time" $-\infty \leq \tau \leq +\infty$

σ - "internal dimension" $0 \leq \sigma \leq \pi$

} world-sheet coordinates.

- The string coordinates: $X^\mu(\tau,\sigma)$ $\mu = 0,1,2 \ldots 9$

target-space.

The world-sheet metric is $\begin{pmatrix} -1 & \\ & +1 \end{pmatrix}$ → Minkowski,

as before we find the element of surface area dA.

by using the tangent vector dv_1^μ dv_2^μ.

with $dv_1^\mu = \dfrac{\partial X^\mu}{\partial \tau} d\tau$. $dv_2^\mu = \dfrac{\partial X^\mu}{\partial \sigma} d\sigma$.

- The only difference is that the sign of the argument in $dA = \sqrt{\quad g \quad}\, d\tau d\sigma$. is reversed.

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⇒ The area functional is defined as.

$$A = \int d\tau\, d\sigma \left[\left(\frac{\partial X^\mu}{\partial \tau} \cdot \frac{\partial X_\mu}{\partial \sigma} \right)^2 - \left(\frac{\partial X^\mu}{\partial \tau} \frac{\partial X_\mu}{\partial \tau} \right) \left(\frac{\partial X^\nu}{\partial \sigma} \frac{\partial X_\nu}{\partial \sigma} \right) \right]^{1/2}$$

The argument under the square is positive definite.

The fact that this argument is positive definite also follows from the fact that we require that the tangent vector space is two dimensional everywhere

general tangent vector $v^\mu(\lambda) = \frac{\partial X^\mu}{\partial \tau} + \lambda \frac{\partial X^\mu}{\partial \sigma}$

$\lambda \in (-\infty, +\infty)$

consider the square.

$$v^\mu(\lambda)\, v_\mu(\lambda) = \lambda^2 \left(\frac{\partial X^\mu}{\partial \sigma} \right)^2 + 2\lambda \left(\frac{\partial X^\mu}{\partial \tau} \cdot \frac{\partial X_\mu}{\partial \sigma} \right) + \left(\frac{\partial X}{\partial \tau} \right)^2.$$

To guarantee the existence of two real solutions the discriminant must be positive

$-\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - ac}$

i.e. $$\left(\frac{\partial X}{\partial \tau} \cdot \frac{\partial X}{\partial \sigma} \right)^2 - \left(\frac{\partial X}{\partial \sigma} \right)^2 \cdot \left(\frac{\partial X}{\partial \tau} \right)^2 > 0$$

which is precisely the condition we wanted to prove.

The Nambu-Gotto string action.

To identify the area functional as the action of the string we need to fix its dimension.

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$[S] = [E][T] = [M]\frac{[L]^2}{[T]^2}[T] = [M]\frac{[L]^2}{[T]}$

Area functional $[A] = [L]^2 \Rightarrow [S] = \frac{[M]}{[T]}[A]$

The string tension has dimension of force. $[F] = \frac{[M][L]}{[T]^2} = \frac{[M][L]}{[T][T]}$

hence a factor with the right dimensions is given by $\frac{T_0}{c}$,

where T_0 — string tension c — speed of light, $\frac{[T_0]}{[c]} = \frac{[M]}{[T]}$

$$\Rightarrow S = -\frac{T_0}{c}\int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}$$

where $\sigma_1 > 0$ is a constant. (e.g. $\sigma_1 = \pi$)

$$\dot{X} = \frac{\partial X^\mu}{\partial \tau} \qquad X' = \frac{\partial X^\mu}{\partial \sigma}$$

S is the Nambu-Gotto action of the relativistic string

As before the action is invariant under reparameterizations.

write $-ds^2 = dX^\mu dX^\mu = \eta_{\mu\nu} dx^\mu dx^\nu = \eta_{\mu\nu}\frac{\partial X^\mu}{\partial \eta^\alpha}\frac{\partial X^\nu}{\partial \eta^\beta} d\eta^\alpha d\eta^\beta$

take $\eta_1 = \tau \qquad \eta_2 = \sigma$

define $\gamma_{\alpha\beta} = \eta_{\mu\nu}\frac{\partial X^\mu}{\partial \eta^\alpha}\frac{\partial X^\nu}{\partial \eta^\beta} = \frac{\partial X}{\partial \eta^\alpha}\cdot\frac{\partial X}{\partial X^\beta} = \begin{bmatrix} (\dot{X})^2 & \dot{X}X' \\ \dot{X}X' & (X')^2 \end{bmatrix}$

Then $S = -\frac{T_0}{c}\int d\tau d\sigma \sqrt{-\gamma}$

where $\gamma = \det(\gamma_{\alpha\beta})$. The proof follows as before.

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Additionally: The Nambu-Goto action is invariant under space-time Poincare transformations. $X'^{\mu} = \Lambda^{\mu}{}_{\nu} X^{\nu} + a^{\mu}$

Equations of motion and boundary conditions

Having established the form of the action we can obtain the Eqs. of motion from the variation of the action under $X \to X + \delta X$

$$S_{NG} = \int_{\tau_i}^{\tau_f} d\tau\, L = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma\, \mathcal{L}(\dot{X}^{\mu}, X'^{\mu}) = \int d^2\sigma\, \mathcal{L}$$

where $\mathcal{L}(\dot{X}^{\mu}, X'^{\mu}) = -\frac{T_0}{c}\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2 (X')^2}$ → Lagrangian density.

$$\delta S = \int d^2\sigma\, \mathcal{L}(\dot{X}^{\mu} + \delta\dot{X}^{\mu}, X'^{\mu} + \delta X'^{\mu}) - \int d^2\sigma\, \mathcal{L}(\dot{X}^{\mu}, X'^{\mu})$$

$$= \cancel{\int d^2\sigma\, \mathcal{L}(\dot{X}^{\mu}, X'^{\mu})} + \int d^2\sigma \left(\frac{\partial\mathcal{L}}{\partial\dot{X}^{\mu}}\delta\dot{X}^{\mu} + \frac{\partial\mathcal{L}}{\partial X'^{\mu}}\delta X'^{\mu} \right) + O(\delta^2) - \cancel{\int d^2\sigma\, \mathcal{L}}$$

$$= \int d^2\sigma\, \frac{\partial\mathcal{L}}{\partial\dot{X}^{\mu}}\frac{\partial\, \delta X^{\mu}}{\partial\tau} + \frac{\partial\mathcal{L}}{\partial X'^{\mu}}\frac{\partial}{\partial\sigma}\delta X^{\mu}$$

Here $\frac{\partial\mathcal{L}}{\partial\dot{X}^{\mu}} = P^{\tau}_{\mu} = \Pi_{\mu} = -\frac{T_0}{c}\frac{(\dot{X}\cdot X')X'_{\mu} - (X')^2\dot{X}_{\mu}}{\left((\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2\right)^{1/2}}$

$$\frac{\partial\mathcal{L}}{\partial X'^{\mu}} = P^{\sigma}_{\mu} = -\frac{T_0}{c}\frac{(\dot{X}\cdot X')\dot{X}_{\mu} - (\dot{X})^2 X'_{\mu}}{\left((\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2\right)^{1/2}}$$

Π_{μ} is the canonical momentum.

using this notation. and integration by parts.

$$\to \delta S' = \int d^2\sigma \left[\frac{\partial}{\partial \tau}\left(P^{\tau}_{\mu}\,\delta X^{\mu}\right) + \frac{\partial}{\partial \sigma}\left(P^{\sigma}_{\mu}\,\delta X^{\mu}\right) - \left(\frac{\partial P^{\tau}_{\mu}}{\partial \tau} + \frac{\partial P^{\sigma}_{\mu}}{\partial \sigma}\right)\delta X^{\mu}\right]$$

The first term is a boundary term in τ.

imposing: $\delta X^{\mu}(\tau_i, \sigma) = \delta X^{\mu}(\tau_f, \sigma) = 0$. it vanishes.

The last term must vanish because δX^{μ} is arbitrary

hence, $\frac{\partial P^{\tau}_{\mu}}{\partial \tau} + \frac{\partial P^{\sigma}_{\mu}}{\partial \sigma} = 0 \Rightarrow$ Eq. of motion of the Relativistic string

we will use the gauge freedom (reparameterization invariance) to simplify

The second term is a boundary term in $\sigma \to$ string end point.

$\tau \to$ not fixed $\to$ $$\int d\tau \left[\frac{\partial \mathcal{L}}{\partial X'^{\mu}}\,\delta X^{\mu}\right]_{\sigma=0}^{\sigma=\sigma_1} \overset{!}{=} 0.$$

For $\delta S' = 0$ this term must vanish.

$\mu = 0, 1, \ldots D-1 \Rightarrow 2D$ boundary conditions

Three possibilities

1) Periodic boundary conditions.

$$X(\sigma_1) = X(\sigma_1 + \pi) \to \text{closed string}$$

The world-sheet does not have boundaries

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2) Neumann boundary conditions.

$$\left.\frac{\partial \mathcal{L}}{\partial X'^{\mu}}\right|_{0,\sigma_1} = 0.$$

also called free boundary conditions
open string ends can move freely
The momentum at the end of the string is conserved.

3) Dirichlet boundary conditions.

$$\delta X^{\mu}(\tau, 0) = \delta X^{\mu}(\tau, \sigma_1) = 0 \qquad \mu \neq 0$$

Open strings with ends kept ~~fix~~ fixed in the i-th directi

For $\mu = 0$ we must have $P_0^{\sigma}(\tau, 0) = P_0^{\sigma}(\tau, \sigma_1) = 0$.

we can have Neumann boundary conditions along time and p space-like coordinates and Dirichlet boundary conditions along D-p directions. The end points (free to move) of the string are fixed on p-dimensional space-like surfaces called Dirichlet p-branes

Example: D2-brane in a the dimensional space is fixed in one direction say $x^3 = 0$ and free to move in the x^1, x^2 directions.

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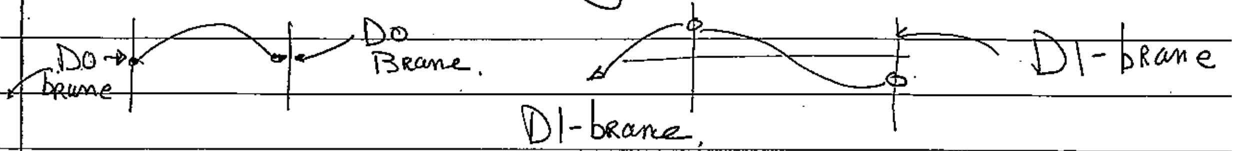
For the nonrelativistic string that we saw before.



D9 or space filling brane - free endpoints in all directions.

The Polyakov Action

As in the case of the point particle working with the Nambu-Gotto action is complicated because of the square root.

We want to a form of the string action which is equivalent to the Nambu-Gotto action, but without the square root → as usual we introduce auxilliary fields, analogous to $e(\tau)$

$h_{\alpha\beta}(\tau, \sigma)$ → world-sheet metric.

$h^{\alpha\beta} = (h^{-1})_{\alpha\beta}$ $\quad h = \det(h_{\alpha\beta})$

→ $$S_\sigma = -\frac{T}{2}\int d^2\sigma \sqrt{-h}\, h^{\alpha\beta} \partial_\alpha X \partial_\beta X$$ ← string sigma model action

At the classical level the string sigma model is equivalent to the Nambu Gotto action.

More convenient for quantization.

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We want to derive the eqs. of motion for $h_{\alpha\beta}$ and X^μ.

The variation of the determinant is given by

$$\delta h = - h\, h_{\alpha\beta}\, \delta h^{\alpha\beta}$$

$$\Rightarrow \quad \delta\sqrt{-h} = -\frac{1}{2}\sqrt{-h}\; h_{\alpha\beta}\,\delta h^{\alpha\beta}$$

$$\Rightarrow \frac{\delta S}{\delta h_{\alpha\beta}} = -\frac{T\sqrt{-h}}{2}\,\partial_\alpha X\cdot\partial_\beta X + \frac{T}{2}\frac{1}{2}\sqrt{-h}\; h_{\alpha\beta}\, h^{\gamma\delta}\,\partial_\gamma X\cdot\partial_\delta X =$$

The world sheet energy momentum tensor is given by

$$T_{\alpha\beta} = -\frac{2}{T}\frac{1}{\sqrt{-h}}\frac{\delta S}{\delta h_{\alpha\beta}}$$

hence $$T_{\alpha\beta} = \partial_\alpha X\cdot\partial_\beta X - \frac{1}{2} h_{\alpha\beta}\, h^{\gamma\delta}\,\partial_\gamma X\cdot\partial_\delta X = 0$$

This is the equation of motion for $h_{\alpha\beta}$, which can used to eliminate $h_{\alpha\beta}$ from the action.

From $T_{\alpha\beta}=0 \Rightarrow \quad \partial_\alpha X\cdot\partial_\beta X = \frac{1}{2} h_{\alpha\beta}\, h^{\gamma\delta}\,\partial_\gamma X\cdot\partial_\delta X$

take the determinant of both sides. $\quad h = \det(h_{\alpha\beta})$

$$\det(\partial_\alpha X\cdot\partial_\beta X) = \frac{1}{2^2}\, h\,(h^{\gamma\delta}\partial_\gamma X\cdot\partial_\delta X)^2$$

multiply by (-1) and take the square root.

$$\sqrt{-\det(\partial_\alpha X\cdot\partial_\beta X)} = \frac{1}{2}\sqrt{-h}\; h^{\gamma\delta}\,\partial_\gamma X\cdot\partial_\delta X$$

on the left-hand side we have the argument of the NG action

on the right " " " " " " " " Polyakov "

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The equation of motion for X^μ is obtained from the Euler - Lagrange equation. $\partial_\alpha \frac{\partial \mathcal{L}}{\partial \partial_\alpha X^\mu} - \frac{\partial \mathcal{L}}{\partial X^\mu} = 0.$

$$\Rightarrow \frac{1}{\sqrt{-h}} \partial_\alpha \left(\sqrt{-h}\, h^{\alpha\beta} \partial_\beta X^\mu \right) = 0.$$

Symmetries of the Polyakov action

local symmetries on the world-sheet Σ with signature $(-1, +1)$

1) Reparameterization $\rightarrow$ $\sigma \rightarrow \tilde{\sigma}(\sigma)$ that acts by

$$\tilde{X}^\mu(\tilde{\sigma}) = X^\mu(\sigma)$$

check $\rightarrow$ $$\tilde{h}_{\alpha\beta}(\tilde{\sigma}) = \frac{\partial \sigma^\gamma}{\partial \tilde{\sigma}^\alpha} \frac{\partial \sigma^\delta}{\partial \tilde{\sigma}^\beta} h_{\gamma\delta}(\sigma)$$

Where σ, $\tilde{\sigma}$ imply $(\tau, \sigma) \rightarrow (\tilde{\tau}, \tilde{\sigma})$.

These transformations leave the action invariant and are also called diffeomorphisms.

2) Weyl-transformations

The action is invariant under the rescaling.

$$h_{\alpha\beta} \rightarrow e^{\phi(\tau,\sigma)} h_{\alpha\beta} \quad \text{and} \quad \delta X^\mu = 0.$$

Recall $h_{\alpha\beta} h^{\alpha\beta} = 1 \Rightarrow h^{\alpha\beta} \rightarrow e^{-\phi} h^{\alpha\beta}$ & $h = \det(h_{\alpha\beta}) \Rightarrow h \rightarrow e^{2\phi} h$

since $\sqrt{-h} \rightarrow e^{+\phi} \sqrt{-h}$ and $h^{\alpha\beta} \rightarrow e^{-\phi} h^{\alpha\beta}$

give cancelling factors in the action.

Remarks 1 A Weyl transformation is not a diffeomorphism.

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It is a multiplication of the metric by a constant factor, usually called a conformal transformation by mathematicians.

2) The invariance under Weyl transformations is special to strings. It does not occur for particles, membranes and higher p-branes.

This is indeed obvious because it arises from the fact that we have taken the $\sqrt{-h} \cdot h^{\alpha\beta}$ where $h_{\alpha\beta}$ is a two dimensional metric.

3) combining Weyl with reparameterization invariance, one has three local transformations which can be used to gauge fix the metric $h_{\alpha\beta}$ completely.

Thus, $h_{\alpha\beta}$ does not introduce new local degrees of freedom. It is an auxiliary (dummy) field.

$\Rightarrow$ The combined reparameterization + Weyl symmetry is the unique and powerful property of the 1-dimensional extended object, i.e. string as we will see later on - preserving these properties at the quantum level is the pivotal requirement in formulating the quantized string.

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Global symmetries with respect to M.

Poincare transformations: $X'^{\mu} = \Lambda^{\mu}{}_{\nu} X^{\nu} + b^{\mu}$

$\Lambda^{\mu}{}_{\nu} \to$ Lorentz transformations $\qquad b^{\mu} \to$ constant S.T. shifts.

gauge fixing

we can use the local world-sheet symmetries (reparameterization & Weyl) to choose a convenient parameterization,

e.g. The static gauge,

In the static gauge we fix the longitudenal directions

$c\tau = X^0 = ct = t \qquad$ (if we set $c=1$).

for any point on the world-sheet in some Lorentz frame we can describe the string coordinates X^{μ} as.

$$X^{\mu}(\tau, \sigma) = X^{\mu}(t, \sigma) = \{ct, \vec{X}(t,\sigma)\},$$

where $\vec{X}$ represents the spatial coordinates.

Then |

$$\frac{\partial X^{\mu}}{\partial \sigma} = \left(\frac{\partial X^0}{\partial \sigma}, \frac{\partial \vec{X}}{\partial \sigma}\right) = \left(0, \frac{\partial \vec{X}}{\partial \sigma}\right)$$

$$\frac{\partial X^{\mu}}{\partial \tau} = \left(\frac{\partial X^0}{\partial t}, \frac{\partial X^{\mu}}{\partial t}\right) = \left(c, \frac{\partial \vec{X}}{\partial t}\right).$$

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So far we used one gauge symmetry.

conformal gauge | next we can use the reparameterization invariances to fix $h_{\alpha\beta} = e^{\phi}\eta_{\alpha\beta}$

where $\eta_{\alpha\beta}$ is the flat metric $\eta_{\alpha\beta} = \begin{pmatrix} -1 & \\ & +1 \end{pmatrix}$.

flat metric | Finally, we can use the Weyl invariance to fix

$$h_{\alpha\beta} = \begin{pmatrix} h_{00} & h_{01} \\ h_{10} & h_{11} \end{pmatrix} = \eta_{\alpha\beta} = \begin{pmatrix} -1 & \\ & +1 \end{pmatrix}.$$

Reparameterization & Weyl invariances are symmetries of the classical string, that in general are spoiled in the quantized string. Restoring these symmetries in the quantized string results in some constraints.

again | <u>The equations of motion and the stress energy tensor</u>

The choice of the flat world-sheet metric

$$h_{\alpha\beta} = \eta_{\alpha\beta} = \begin{pmatrix} -1 & \\ & +1 \end{pmatrix}.$$

simplifies the eqs. of motion.

The Polyakov action becomes:

$$S_P = -\frac{T}{2}\int d^2\sigma\, \eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu$$

$$= -\frac{T}{2}\int d^2\sigma\left(-\dot{X}^2 + X'^2\right).$$

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The equations of motion for X^μ becomes:

$$\Box X^\mu = \partial_\alpha \partial^\alpha X^\mu(\tau, \sigma) = 0.$$

OR:

$$\left(-\frac{\partial^2}{\partial \tau^2} + \frac{\partial^2}{\partial \sigma^2}\right) X^\mu(\tau, \sigma) = 0. \quad (*)$$

We must also impose the constraints from variation with respect to $h_{\alpha\beta}$.

$$T_{\alpha\beta} = -\frac{2}{T}\frac{\delta S}{\delta h^{\alpha\beta}} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta X_\mu$$

setting $h_{\alpha\beta} = \eta_{\alpha\beta}$ gives

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}\eta_{\alpha\beta}\eta^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta X_\mu.$$

We saw: The equation of motion associated with $h_{\alpha\beta} \Rightarrow T_{\alpha\beta} = 0$

OR: $T_{01} = \frac{\partial X^\mu}{\partial \tau} \cdot \frac{\partial X_\mu}{\partial \sigma} - \frac{1}{2}(0) = 0 \Rightarrow \dot{X} \cdot X' = 0.$

$\boxed{X^2 = X^\mu X_\mu}$

$$T_{00} = +\dot{X}^2 - \frac{1}{2}(-1)\cdot(-\dot{X}^2 + X'^2) = \frac{1}{2}(\dot{X}^2 + X'^2) = 0$$

$$T_{11} = X'^2 - \frac{1}{2}(+1)(-\dot{X}^2 + X'^2) = \frac{1}{2}(\dot{X}^2 + X'^2) = 0.$$

We see: The equation of motion of the string are the free wave equation (*) subject to the constraints (**) that arise from the equation of motion $T_{\alpha\beta} = 0$.

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what is the meaning of the constraints?

$T_{01} = 0$ $\dot{X} \cdot X' = 0$. $\rightarrow$ Parameterization τ, σ such that line of constant σ are perpendicular to lines of constant τ.

take the static gauge with $X^0 = t = R\tau$

then $(X^0)' = 0$ $\dot{X}^0 = R$ $R \rightarrow$ constant.

then $\partial_\alpha \partial^\alpha X^\mu = 0 \Rightarrow \ddot{\vec{X}} - \vec{X}'' = 0$ where $X^\mu = (X^0, \vec{X})$

and the constraint become

$$\dot{\vec{X}} \cdot \vec{X}' = 0$$

$$\dot{\vec{X}}^2 + \vec{X}'^2 = R^2$$

The first constraint means that the motion of the string must be perpendicular to the string itself.

i.e. the physical modes of the string are transverse oscillations.

There is no longitudinal mode.

The constant R is related to the length of the string when $\dot{\vec{X}} = 0$

Also | Note $Tr\, T = \eta^{\alpha\beta} T_{\alpha\beta} = -T_{00} + T_{11} = 0$

$\rightarrow$ a consequence of weyl invariance.

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Reminder | boundary conditions are similar to those that we encountered in the nonrelativistic case and arise from variation of the action. $X^\mu \to X^\mu + \delta X^\mu$.

fix $0 \leq \sigma \leq \pi$

In addition to the eq. of motion there is a boundary term. $-T \int d\tau \left[X'_\mu \delta X^\mu \Big|_{\sigma=\pi} - X'_\mu \delta X^\mu \Big|_{\sigma=0} \right]$

that must vanish.

* closed string $X^\mu(\sigma,\tau) = X^\mu(\sigma+\pi, \tau)$.

* open string with Neumann boundary conditions.

$$\frac{\partial X^\mu}{\partial \sigma} = 0 \quad \text{at} \quad \sigma = 0, \pi$$

No momentum is flowing through the ends of the string.

* open string with Dirichlet boundary conditions.

The position of the two ends are fixed $\to \delta X^\mu = 0$

$$X^\mu \Big|_{\sigma=0} = X_0^\mu \qquad X^\mu \Big|_{\sigma=\pi} = X_\pi^\mu.$$

where X_0^μ, X_π^μ are constants and $\mu = 1, \dots D-p-1$

Dirichlet boundary conditions break Poincare invariance

We will impose the boundary conditions on solutions of the Eq. of motion resulting in different strings.

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Solutions of the eq. of motion and light-cone coordinates

The eq. of motion is a two dimensional wave equation.

$$\partial_\alpha \partial^\alpha X^\mu(\tau, \sigma) = -\frac{\partial^2 X^\mu}{\partial \tau^2} + \frac{\partial^2 X^\mu}{\partial \sigma^2} = 0.$$

and has the solutions:

$$X^\mu(\tau, \sigma) = X_L^\mu(\tau + \sigma) + X_R^\mu(\tau - \sigma)$$

corresponding to left- and right-moving waves.

The form of the solutions suggests the introduction of light-cone coordinates:

$$\sigma^\pm := \sigma^0 \pm \sigma^1 = \tau \pm \sigma.$$

we write: σ^a where $a = +, -$ for L.C. coordinat

σ^α where $\alpha = 0, 1$ for standard coordinat

The light-cone coordinates facilitates the quantization of the string → light-cone quantization (R/NS).

The alternative is covariant quantization (G/S) and is beyond the scope of these lectures.

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- The Jacobian of the coordinate transformation and its inverse: $\left(\sigma^0 = \frac{\sigma^+ + \sigma^-}{2}, \ \sigma^1 = \frac{\sigma^+ - \sigma^-}{2}\right)$

$$\left(J_\alpha{}^a\right) = \frac{D(\sigma^+, \sigma^-)}{D(\sigma^0, \sigma^1)} = \begin{pmatrix} \frac{\partial\sigma^+}{\partial\sigma^0} & \frac{\partial\sigma^+}{\partial\sigma^1} \\ \frac{\partial\sigma^-}{\partial\sigma^0} & \frac{\partial\sigma^-}{\partial\sigma^1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

$$\left(J_a{}^\alpha\right) = \frac{D(\sigma^0, \sigma^1)}{D(\sigma^+, \sigma^-)} = \begin{pmatrix} \frac{\partial\sigma^0}{\partial\sigma^+} & \frac{\partial\sigma^0}{\partial\sigma^-} \\ \frac{\partial\sigma^1}{\partial\sigma^+} & \frac{\partial\sigma^1}{\partial\sigma^-} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

- converting lower indices:

$$V_a = J_a{}^\alpha V_\alpha, \qquad V_\alpha = J_\alpha{}^a V_a$$

Example: The light-cone derivatives.

$$\begin{pmatrix} \partial_+ \\ \partial_- \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \partial_0 \\ \partial_1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \partial_0 + \partial_1 \\ \partial_0 - \partial_1 \end{pmatrix}$$

Check: $\frac{\partial}{\partial\sigma^+} = \frac{\partial\sigma^0}{\partial\sigma^+}\frac{\partial}{\partial\sigma^0} + \frac{\partial\sigma^1}{\partial\sigma^+}\frac{\partial}{\partial\sigma^1} = \frac{1}{2}\frac{\partial}{\partial\sigma^0} + \frac{1}{2}\frac{\partial}{\partial\sigma^1}$

- $\frac{\partial}{\partial\sigma^-} = \frac{\partial\sigma^0}{\partial\sigma^-}\frac{\partial}{\partial\sigma^0} + \frac{\partial\sigma^1}{\partial\sigma^-}\frac{\partial}{\partial\sigma^1} = \frac{1}{2}\frac{\partial}{\partial\sigma^0} - \frac{1}{2}\frac{\partial}{\partial\sigma^1}$

same ✓

converting upper indices:

$$W^a = W^\alpha J_\alpha{}^a, \qquad W^\alpha = W^a J_a{}^\alpha$$

Example: The light-cone differentials.

- $$(d\sigma^+, d\sigma^-) = (d\sigma^0, d\sigma^1)\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = (d\sigma^0 + d\sigma^1, d\sigma^0 - d\sigma^1)$$

✓

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when converting tensors, act on each index.

Example: $h_{ab} = J_a{}^{\alpha} J_b{}^{\beta} h_{\alpha\beta}$

Explicitly: metric in standard coordinates:

$$h_{\alpha\beta} = \begin{pmatrix} -1 & \\ & +1 \end{pmatrix} = h^{\alpha\beta}$$

metric in light-cone coordinates:

$$h_{++} = J_+{}^{\alpha} J_+{}^{\beta} h_{\alpha\beta} = -J_+{}^0 J_+{}^0 + J_+{}^1 J_+{}^1 = -\tfrac{1}{4} + \tfrac{1}{4} = 0 = h_{--}$$

$$h_{+-} = J_+{}^{\alpha} J_-{}^{\beta} h_{\alpha\beta} = -J_+{}^0 J_-{}^0 + J_+{}^1 J_-{}^1 = -\tfrac{1}{4} - \tfrac{1}{4} = -\tfrac{1}{2} = h_{-+}$$

hence $(h_{ab}) = \begin{pmatrix} h_{++} & h_{+-} \\ h_{-+} & h_{--} \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

$$h^{ab} = (h_{ab})^{-1} = -2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The symmetric and traceless energy momentum tensor

(i.e. $T_{00} = T_{11}$ and $T_{01} = T_{10}$).

$$T_{++} = J_+{}^{\alpha} J_+{}^{\beta} T_{\alpha\beta} = \frac{1}{4}(T_{00} + T_{01})\cdot 2 = \frac{1}{2}(T_{00} + T_{01})$$

$$T_{--} = J_-{}^{\alpha} J_-{}^{\beta} T_{\alpha\beta} = \frac{1}{4}(T_{00} - T_{01})\cdot 2 = \frac{1}{2}(T_{00} - T_{01}).$$

$$T_{+-} = J_+{}^{\alpha} J_-{}^{\beta} T_{\alpha\beta} = \frac{1}{4}(T_{00} - T_{01} + T_{10} - T_{11}) = 0 = T_{-+}.$$

Note The trace in the light cone coordinates is.

$$\text{Trace } T = \eta^{ab} T_{ab} = 2\eta^{+-} T_{+-} = -4\, T_{+-}$$

Thus, $\mathrm{Tr}\, T = 0 \iff T_{+-} = 0$.

The action in light-cone coordinates takes the form.

$$S = -\frac{T}{2}\int d^2\sigma\, \eta^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X_\mu =$$

$$= \frac{T}{2}\int d^2\sigma\, (\dot{X}^2 - X'^2) = \ ?$$

<u>From</u> $\partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1) \longrightarrow \partial_0 = (\partial_+ + \partial_-); \partial_1 = (\partial_+ - \partial_-)$

$$\Rightarrow \left(\frac{\partial X}{\partial \tau}\right)^2 - \left(\frac{\partial X}{\partial \sigma}\right)^2 = (\partial_+ X + \partial_- X)^2 - (\partial_+ X - \partial_- X)^2 =$$

$$= 4\partial_+ X \cdot \partial_- X$$

$$\Rightarrow S' = 2T\int d^2\sigma\, \partial_+ X^\mu \partial_- X_\mu.$$

The Equations of motion in light cone coordinates.

$$\Box X^\mu = -(\partial_0^2 - \partial_1^2)\, X^\mu = -4\partial_+\partial_- X^\mu = 0.$$

<u>That is:</u> $\partial_+ \partial_- X^\mu = 0$

It is obvious that the general solution is.

$$X^\mu(\sigma^I) = X_L^\mu(\sigma^0 + \sigma^1) + X_R^\mu(\sigma^0 - \sigma^1)$$

The constraints in light cone coordinates:

$$T_{++} = \frac{1}{2}(T_{00} + T_{01}) = \frac{1}{2}\left(\frac{1}{2}(\dot{X}^2 + X'^2) + \dot{X}\cdot X'\right)$$

$$= \frac{1}{4}(\dot{X} + X')^2 = \left[\frac{1}{2}\left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\sigma}\right)X\right]^2 = \partial_+ X\, \partial_+ X$$

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hence $T_{++} = 0 \iff \partial_+ X^\mu \partial_+ X_\mu = 0 \iff \dot{X}_L^2 = 0$

why.

similarly $T_{--} = 0 \iff \partial_- X^\mu \partial_- X_\mu = 0 \iff \dot{X}_R^2 = 0$

The condition $T_{+-} = 0$ is not listed as a constraint because it holds off-shell.

The two dimensional energy momentum tensor satisfies a conservation equation.

$$\partial^\alpha T_{\alpha\beta} = 0.$$

In light coordinate these become:

$\partial_- T_{++} = 0$ i.e. $T_{++} = T_{++}(\sigma^+)$.

$\partial_+ T_{--} = 0$ i.e. $T_{--} = T_{--}(\sigma^-)$.

Using this we note that every function $f(\sigma^+)$ defines a conserved chiral current.

as $\partial_- \left(f(\sigma^+) T_{++}\right) = 0$,

and Therefore a conserved charge.

$$L_f = T \int_0^\pi d\sigma' \, f(\sigma^+) \, T_{++}$$

and similarly for T_{--} with $f(\sigma^-)$.