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where $vol(S^{D-1})$ is a volume of $vol(S^{D-1}(R))$ with R =

we need to find the value of $vol(S^{D-1})$.

we can do this by evaluating the integral

$$I_D = \int_{R^D} dx_1 \dots dx_d \, e^{-r^2}$$

where $x_1^2 + \dots + x_D^2 = r^2$.

in two ways.

1) $$I_D = \int_{-R^D}^{\infty} dx_1 \cdots dx_D \, e^{-x_1^2 + \dots + x_D^2} = \prod_{i=1}^{D} \int_{-\infty}^{\infty} dx_i \, e^{-x_i^2} = (\sqrt{\pi})^d = \pi$$

2) $$I_D = \int_0^\infty dr \, Vol(S^{D-1}(r)) e^{-r^2} = Vol(S^{D-1}) \int_0^\infty dr \, r^{D-1} e^{-r}$$

substitute $r^2 = t$ $\quad r\,dr = \frac{dt}{2}$ $\quad r = t^{\frac{1}{2}}$. $\quad r^{D-2} = t^{\frac{D-2}{2}} = t^{\frac{D}{2}-1}$

$$\Rightarrow I_D = Vol(S^{D-1}) \cdot \int_0^\infty \frac{dt}{2} \, t^{\frac{D}{2}-1} e^{-t}.$$

the last integral can be expressed in terms of the gamma function.

$$\Gamma(x) = \int_0^\infty dt \, e^{-t} \, t^{x-1}, \qquad x > 0$$

$$\Rightarrow I_D = \frac{1}{2} Vol(S^{D-1}) \, \Gamma\left(\frac{D}{2}\right).$$

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equating the two results we get.

$$\mathrm{vol}(S^{D-1}) = \frac{2\pi^{\frac{D}{2}}}{\Gamma(D/2)}$$

what is $\Gamma(D/2)$? D = integer.

$$\Gamma(x+1) = \int_0^\infty dt\, e^{-t} t^{(x+1)-1} = \int_0^\infty dt\, e^{-t} t^{x} =$$

$$= -\int_0^\infty dt \frac{d(e^{-t})}{dt} t^x = -\left[e^{-t} x t^{x-1}\right]_0^\infty - x\int_0^\infty dt\, e^{-t} t^{x-1}$$

$$= x\int_0^\infty dt\, e^{-t} t^{x-1} = x\,\Gamma(x)$$

so $\Gamma(x+1) = x\,\Gamma(x)$

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty dt\, e^{-t} t^{\frac{1}{2}-1} = \int_0^\infty dt\, e^{-t} t^{-\frac{1}{2}} = \int_0^\infty du\, e^{-u^2}\cdot 2 = \sqrt{\pi}$$

$t = u^2 \quad dt = 2u\,du$

$$\Gamma(1) = \int_0^\infty dt\, e^{-t} t^{1-1} = -e^{-t}\Big|_0^\infty = \underline{1}.$$

hence for x = integer. $\Gamma(n+1) = n\Gamma(n) = n!$

$$\Gamma(3/2) = \Gamma(\tfrac{1}{2}+1) = \tfrac{1}{2}\Gamma(\tfrac{1}{2}) = \tfrac{1}{2}\sqrt{\pi} \;/\; \Gamma(5/2) = \tfrac{3}{4}\sqrt{\pi}.$$

so $$\mathrm{vol}(S^3) = \mathrm{vol}(S^{4-1}) = \frac{2\cdot\pi^{\frac{4}{2}}}{\Gamma(4/2)} = 2\pi^2 \ldots$$

<u>electric fields in higher dimensions</u>

Electric field of a point charge in D-spatial dimensions

Electric field → Radial → similar to gravitational field.

use Maxwell's eq. in higher D

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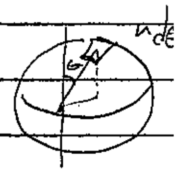
start with: $\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \frac{1}{c} j^\mu$ → Maxwell eq. in the presence of source

since $F^{0i} = E_i$ This equation is Gauss law.

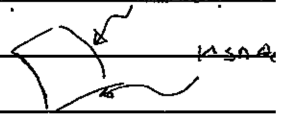
$\vec{\nabla}\cdot\vec{E} = \rho$ → valid in any dimension → find E^i $i = 1 - \bar{1}$

Gauss law in 3D: $q_{enclosed} = \int_{B^3} d^3x\, \rho(\vec{x}) = \int_{B^3} d^3x\, \vec{\nabla}\cdot\vec{E} = \int_{S^2} \vec{E}\cdot d\vec{A}$



The volume B^3 is enclosed by a sphere S^2

The electric field is constant on the surface of the sphere and is in the radial direction. E_r $r d\theta$



$dA = r\sin\theta\, d\phi \cdot r\, d\theta$

$$\int_{S^2} \vec{E}\cdot d\vec{A} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \cdot r^2 \sin\theta\, E_r =$$

$$= E_r \cdot r^2 \cdot 4\pi = q_{enclosed} \Rightarrow E(r) = \frac{q}{4\pi r^2}.$$

The divergence theorem holds in D-dimensions:

$$\int_{V^d} d^n x\, \vec{\nabla}_D\cdot\vec{E} = \text{Flux of } \vec{E} \text{ across } \partial V^d = \int_{\partial V^d} \vec{E}\cdot d\vec{S}$$

Gauss' Law in higher D: $q = \int_{B^D} d^n x\, \rho(x) = \int_{B^D} d^n x\, \vec{\nabla}\cdot\vec{E} = \int_{S^{D-1}} \vec{E}\cdot d\vec{S}$.

For $\vec{E} = E(r)\,\hat{r}$ ⇒ Flux of $\vec{E}$ across $S^{D-1}(r) = q$

$$\Rightarrow \int_{S^{D-1}} \vec{E}\cdot d\vec{S} = E_r \int_{S^{D-1}} dS = E_r \cdot \frac{2\pi^{D/2}}{\Gamma(D/2)} \cdot r^{D-1} = q$$

$$\Rightarrow E_r = \frac{\Gamma(D/2)}{2\pi^{D/2}} \cdot \frac{q}{r^{D-1}}$$

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Gravitation and Planck length

So far | Minkowski space-time $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ $\eta^{\mu\nu} = (-1, +1, +$
Flat space $\eta^{\mu\nu} \to$ constant.

Einstein GR | $\eta_{\mu\nu} \to g_{\mu\nu}(x)$ $ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$

$g_{\mu\nu}(x) \to$ nontrivial function of space-time.

Properties | $g_{\mu\nu}(x) = g_{\nu\mu}(x)$

$g^{\mu\alpha}(x) \, g_{\alpha\nu}(x) = \delta^\mu{}_\nu$

For weak gravitational field: $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$

$\to$ $h_{\mu\nu}(x) \to$ small fluctuation around the Minkowski metric

Einstein eqs. without sources | For $h_{\mu\nu}$: $\partial^2 h^{\mu\nu} - \partial_\alpha (\partial^\mu h^{\nu\alpha} + \partial^\nu h^{\mu\alpha}) + \partial^\mu \partial^\nu h = 0$

where $h^{\mu\nu} = \eta^{\mu\alpha} \eta^{\nu\beta} h_{\alpha\beta}$;

$h = \eta^{\mu\nu} h_{\mu\nu} = -h_{00} + h_{11} + h_{22} + h_{33}$

$\to$ analog of Maxwell equation for fields without sources.

valid only for weak gravitational fields.

The linearized Einstein equation is invariant under the gravitationa

gauge transformations $\delta h^{\mu\nu} = \partial^\mu \epsilon^\nu + \partial^\nu \epsilon^\mu$

where $x'^\mu = x^\mu + \epsilon^\mu(x)$

are infinitesimal changes of the coordinates.

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Newtonian Gravity → non Relativistic weak field limit.

→ Good approximation for many relevant cases.

Newton Law: $|F^{(4)}| = G \frac{m_1 m_2}{r^2}$ $\qquad m_1 \overset{r}{\text{———}} m_2$

$G \to$ four dimensional Newton constant.

Dimensions. $[G] = \frac{[F][L]^2}{[M]^2} = \frac{[M][L]}{[T]^2} \cdot \frac{[L]^2}{[M]^2} = \frac{L^3}{MT^2}$

$G = 6.67 \cdot 10^{-11} \frac{m^3}{kg\, s^2}$

together with $c = 3 \cdot 10^8 \frac{m}{s}$ and $\hbar = 1.06 \times 10^{-34} \frac{kg\, m^2}{s}$.

Gravity: "Planckian" system of units

3 basic units: length, time, mass.

set l_p, t_p, m_p.

such that $G = 1 \cdot \frac{l_p^3}{m_p^2 \cdot t_p^2}$ $\quad c = 1 \cdot \frac{l_p}{t_p}$ $\quad \hbar = m_p \frac{l_p^2}{t_p}$

Solve for l_p, t_p and m_p in terms of $G, c, \hbar$.

$\Rightarrow$ $l_p = \sqrt{\frac{G\hbar}{c^3}} = 1.61 \cdot 10^{-33}$ cm.

$t_p = \frac{l_p}{c} = \sqrt{\frac{G\hbar}{c^5}} = 5.44 \cdot 10^{-44}$ s

$m_p = \frac{\hbar\, t_p}{l_p^2} = 2.17 \cdot 10^{-5}$ g $\simeq 10^{19}$ GeV. ($m_e \sim 0.5$ MeV, $m_p \sim 1$ GeV)

→ Scales of relativistic quantum gravity effects

→ Scales at which Einstein gravity is invalid → strings?

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Non Relativistic strings

mass & tension ; Longitudinal & transversal ; EoM. & Lagrangian.

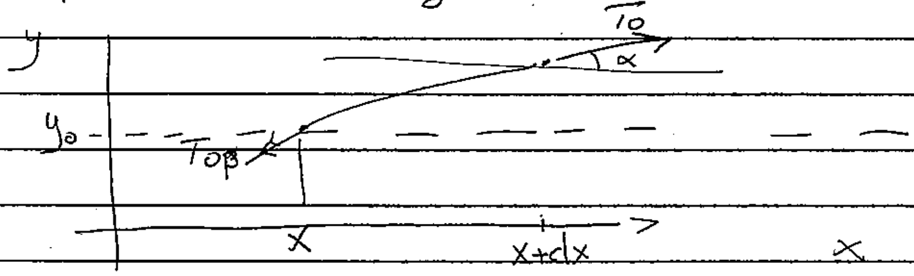
Equations of motion for transverse oscillations

high school mechanics

consider a string (Rope) stretched between two points

we want to write Newton eq. for a segment of the string which is displaced from $y=y_0$.

we want to write down an equation for the propagation of this displacement along the string



T_0 - tension along the string

μ_0 - mass density $= \frac{M}{L}$. M - total mass, L - total length

the net force in the vertical direction is given by

$$T_0 \sin\alpha - T_0 \sin\beta \simeq T_0 \frac{\partial y}{\partial x}\Big|_{x+dx} - T_0 \frac{\partial y}{\partial x}\Big| = T_0 \frac{\partial^2 y}{\partial x^2} dx.$$

for small displacements $\sin\alpha \sim \tan\alpha \sim \frac{\partial y}{\partial x} \ll 1$

hence

$$\mu_0 dx \frac{\partial^2 y}{\partial t^2} = T_0 \frac{\partial^2 y}{\partial x^2} dx$$

OR

$$\frac{\partial^2 y}{\partial x^2} - \frac{\mu_0}{T_0}\frac{\partial^2 y}{\partial t^2} = 0 \Rightarrow \frac{\partial^2 y}{\partial x^2} - \frac{1}{v_0^2}\frac{\partial^2 y}{\partial x^2} = 0$$

→ wave equation for the propagation of the vertical displacement along the string with

$v_0 = \sqrt{T_0/\mu_0}$ - wave velocity.

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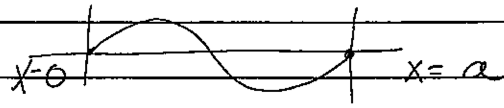
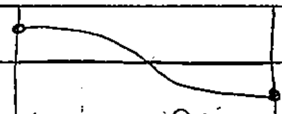
Boundary conditions and initial conditions (space & time).

Boundary conditions

1) Dirichlet : $y(t, x=0) = y(t, x=a) = 0$



Neumann : $\frac{\partial y}{\partial x}(t, x=0) = \frac{\partial y}{\partial x}(t, x=a) = 0.$



The endpoints are free to move in the y-direction.

The general solution of the wave eq. is of the form.

$$y(t, x) = h_+(x - v_0 t) + h_-(x + v_0 t)$$

where h_+, h_- are arbitrary single variable functions.

$h_+ \rightarrow$ Right-moving wave. $h_- \rightarrow$ left-moving.

For a sinusoidal wave.

$$y(t, x) = y(x) \cdot \sin(\omega t + \phi)$$

ω - angular frequency of oscillation.

ϕ - common phase.

what are the allowed frequencies of oscillations?

$$\frac{\partial^2 y}{\partial t^2}(t, x) = -y(x)\,\omega^2 \sin(\omega t + \phi)$$

$$\frac{\partial^2 y}{\partial x^2}(t, x) = y''(x) \sin(\omega t + \phi)$$

substituting into the wave eq.

$$\Rightarrow y''(x) + \frac{w^2}{v_0^2} y(x) = 0 \qquad v_0 = \sqrt{\frac{T_0}{\mu_0}}$$

general solution $y_n(x) = A_n \sin k_n x + B_n \cos k_n x$

FOR Dirichlet boundary conditions $y_n(0) = y_n(a) = 0$

$y_n(0) = B_n \Rightarrow B_n = 0$

$y_n(a) = A_n \sin k_n a = 0 \Rightarrow k_n a = n\pi \Rightarrow k_n = \frac{n\pi}{a}$

$\Rightarrow y_n(x) = A_n \sin \frac{n\pi x}{a} \qquad n = 1, 2, \dots \qquad \boxed{n \neq 0}$

$\Rightarrow \omega_n = \sqrt{\frac{T_0}{\mu_0}} \left(\frac{n\pi}{a}\right) \leftarrow$ from the wave eq.

FOR Neuman boundary conditions $\left.\frac{\partial y}{\partial x}\right|_{x=0} = \left.\frac{\partial y}{\partial x}\right|_{x=a} = 0$

$\frac{\partial y_n(x)}{\partial x} = +A_n k_n \cos k_n x - B_n k_n \sin k_n x$

$y'(0) = A_n \Rightarrow A_n = 0$

$y'(a) = -B_n k_n \sin k_n a = 0 \Rightarrow k_n = \frac{n\pi}{a}$

$\Rightarrow y_n(x) = B_n \cos \frac{n\pi}{a} x \qquad n = 0, 1, 2 \dots$

allowed $y(t,x) = at + b$

A brief Review of Lagrangian dynamics

$L = T - V$ T: kinetic energy
V: potential energy

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For a non relativistic point particle

$L(t) = \frac{1}{2} m (\dot{x}(t))^2 - V(x(t))$ $\quad \dot{x}(t) = \frac{d}{dt} x(t).$

$L(t)$ is an implicit function of time

if $L(t)$ does not depend explicitly on time $\Rightarrow \frac{\partial L}{\partial t} = 0$

Action $\quad S = \int_P L(t)\, dt.$ $\quad P \to$ path of $x(t)$ from x_i to x_f with $t_f > t_i$.

$$S(x) = \int_{t_i}^{t_f} \left(\frac{1}{2} m (\dot{x}(t))^2 - V(x(t)) \right) dt. = S(x)$$

Hamilton Principle: the physical (classical) path correspond to the stationary path $\delta S = 0$.

more precisely: keeping the end points fixed.

$$\delta x(t_i) = \delta x(t_f) = 0$$

and perturbing the path $x(t) \to x(t) + \delta x(t)$

to first order for arbitrary $\delta(x(t))$

$$\Rightarrow S(x + \delta x) = \int_{t_i}^{t_f} \frac{m}{2} \Big(\underbrace{\frac{d}{dt}(x + \delta(x))}_{(\dot{x} + \frac{d}{dt}\delta(x))^2} \Big)^2 - V(x + \delta x) \Big) dt$$

$$= \int_{t_i}^{t_f} \left(\frac{m}{2} \dot{x}^2 - V(x) \right) dt + \left(m \dot{x} \frac{d}{dt} \delta x(t) - V'(x) \delta x(t) \right) dt + O(\delta^2)$$

$$= S(x) + \cancel{m \dot{x} \delta x(t)} \Big|_{t_i}^{t_f} + \int_{t_i}^{t_f} (-m \ddot{x} - V'(x)) \delta x(t)\, dt$$

0

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$$\Rightarrow \delta S(x(t)) = \int_{t_i}^{t_f} \left(-m\ddot{x}(t) - V'(x)\right) \delta(x(t))\, dt.$$

$$\Rightarrow -m\ddot{X}(t) - V'(X) = 0 \quad \text{OR} \quad m\ddot{X}(t) = -V'(X(t)) \to \text{Newton Eq.}$$

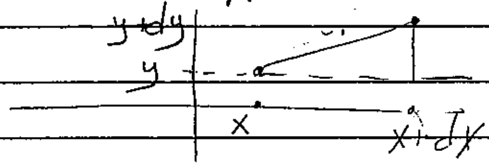
<u>The nonrelativistic string Lagrangian</u> ~~of a~~

mass of a string segment $\mu_0 dx$

$$\Rightarrow T = \int_0^a \frac{1}{2}(\mu_0\, dx)\left(\frac{\partial y}{\partial t}\right)^2 \leftarrow \text{total kinetic energy of the string}$$

Potential energy - work applied to stretch the segments.

Consider



$$\Delta l = \sqrt{dx^2+dy^2} - dx = dx\left(\sqrt{1+\left(\frac{dy}{dx}\right)^2} - 1\right) \simeq dx\,\frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2$$

$\Delta l \to$ change in the length of the string

Work done on segment $= T_0 \Delta l$

$$\Rightarrow V = \int_0^a \frac{1}{2} T_0 \left(\frac{\partial y}{\partial x}\right)^2 dx$$

$$\Rightarrow L(t) = \int_0^a \left(\frac{1}{2}\mu_0\left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2}T_0\left(\frac{\partial y}{\partial x}\right)^2\right) dx = \int_0^a \mathcal{L}\, dx$$

Lagrangian density $\mathcal{L}(x) = \frac{1}{2}\mu_0\left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2}T_0\left(\frac{\partial y}{\partial x}\right)^2$

The action $S = \int_{t_i}^{t_f} L(t)\, dt = \int_{t_i}^{t_f} dt \int_0^a dx \left(\frac{1}{2}\mu_0\left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2}T_0\left(\frac{\partial y}{\partial x}\right)^2\right)$

Here the path is given by $y(t,x) \to$ a field

t, x are implicite variables.

as before variation of the action $\to$ Euler Lagrange E.O.M.

$$y(t,x) \to y(t,x) + \delta y(t,x)$$

$$S(y+\delta y) = \int_{t_1}^{t_2} dt \int_0^a dx \left(\frac{1}{2}\mu_0 \left(\frac{\partial}{\partial t}(y+\delta y)\right)^2 - \frac{1}{2} T_0 \left(\frac{\partial}{\partial x}(y+\delta y)\right)^2 \right)$$

$$= \int_{t_1}^{t_2} dt \int_0^a dx \left(\frac{1}{2}\mu_0 \left(\frac{\partial y}{\partial t}\right)^2 + \frac{\partial y}{\partial t}\frac{\partial}{\partial t}\delta y - \frac{1}{2} T_0 \left(\frac{\partial y}{\partial x}\right)^2 - T_0 \frac{\partial y}{\partial x}\frac{\partial}{\partial x}\delta y \right)$$

$$+ O(\delta^2)$$

$$\Rightarrow \delta S = S(y+\delta y) - S(y) = \int_{t_1}^{t_2} dt \int_0^a dx \left(\mu_0 \frac{\partial y}{\partial t}\frac{\partial \delta y}{\partial t} - T_0 \frac{\partial y}{\partial x}\frac{\partial \delta y}{\partial x} \right).$$

We require that the variation vanishes.

Integrate by parts.

$$\Rightarrow \quad \delta S = \int_{t_i}^{t_f} dt \int_0^a dx \left[\frac{\partial}{\partial t}\left(\mu_0 \frac{\partial y}{\partial t}\delta y\right) - \mu_0 \frac{\partial^2 y}{\partial t^2}\delta y - \frac{\partial}{\partial x}\left(T_0 \frac{\partial y}{\partial x}\delta y\right) + T_0 \frac{\partial^2 y}{\partial x^2}\delta y \right]$$

$$= \int_0^a dx \left(\mu_0 \frac{\partial y}{\partial t}\delta y \right)\Bigg|_{t_i}^{t_f} + \int_{t_i}^{t_f} dt \left(-T_0 \frac{\partial y}{\partial x}\delta y \right)\Bigg|_{x=0}^{x=a}$$

$$- \int_{t_i}^{t} dt \int_0^a dx \left(\mu_0 \frac{\partial^2 y}{\partial t^2} - T_0 \frac{\partial^2 y}{\partial x^2} \right)\delta y.$$

Each term should vanish independently.

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In the third term δy is not restricted

$$\Rightarrow \quad \mu_0 \frac{\partial^2 y}{\partial t^2} - T_0 \frac{\partial^2 y}{\partial x^2} = 0 \rightarrow \text{wave equation.}$$

In the first term we set $\delta y(t_i, x) = \delta y(t_f, x) = 0$

String is fixed at the initial and final time.

Second term $\rightarrow \int dt \left(\frac{\partial y}{\partial x}(t,0)\, \delta y(t,0) - \frac{\partial y}{\partial x}(t,a)\, \delta y(t,a) \right)$

$\delta y(t,0) = \delta y(t,a) = 0 \rightarrow$ Dirichlet B.C.

OR $\frac{\partial y}{\partial x}(t,0) = \frac{\partial y}{\partial x}(t,a) = 0 \rightarrow$ Neumann B.C.

we can also write the Dirichlet B.C. in the form.

$$\frac{\partial y}{\partial t}(t,0) = \frac{\partial y}{\partial t}(t,a) = 0.$$

LST.37 | 24/9/09. 2 | Liverpool |

The relativistic point Particle.

We want to write down the action for a relativistic Point particle in a way that we can generalize for a relativistic string.

For a non-relativistic free particle

$$S_{NR} = \int L_{NR}\, dt = \int \frac{1}{2} m |\dot{\vec{X}}(t)|^2\, dt$$

E.O.M $\quad \frac{d}{dt}\frac{\partial L}{\partial \dot{X}_i} - \frac{\partial L}{\partial X_i} = 0 \Rightarrow \frac{d}{dt}\dot{X} = 0.$

$\Rightarrow \quad \dot{\vec{X}} = \text{const.} \Rightarrow$ not good Relativistically. $\dot{\vec{X}}$ is not bound

To write down the action of a relativistic point Particle we integrate over its world-line.

$$S = -mc^2 \underbrace{\int_P d\tau}_{\text{Lorentz scalar}} = -mc^2 \int_P \frac{ds}{c} = -mc\int ds$$

$$= -mc\int_{t_i}^{t_f} c\,dt\,\sqrt{1-v^2/c^2} = -mc^2\int_{t_i}^{t_f} dt\,\sqrt{1-v^2/c^2}$$

$$= \int dt\; L(\dot{\vec{X}}(t), \vec{X}(t)).$$

$$\Rightarrow \quad L(\dot{\vec{X}}(t), \vec{X}(t)) = -mc^2\sqrt{1-v^2/c^2}$$

For $v > c$ $\quad L$ is not physical

For $v \lll c$ $\quad L \sim -mc^2\left(1 - \frac{1}{2}\frac{v^2}{c^2}\right) = \underbrace{-mc^2}_{\text{ignore} \to \text{constant}} + \frac{1}{2}mv^2$

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The canonical momentum is the derivative of the Lagrangian with respect to $\dot{x}_i$. | $\vec{v}^2 = \sum_i \dot{x}_i^2$

$$\Pi_i = \frac{\partial L}{\partial \dot{x}_i} = \frac{\partial}{\partial \dot{x}_i}\left(-mc^2\sqrt{1-\sum_i \dot{x}_i^2/c^2}\right) = -mc^2 \frac{-2\dot{x}_i/c^2}{2\sqrt{1-\vec{v}^2/c^2}} = \frac{m\dot{x}_i}{\sqrt{1-v^2/c^2}}$$

→ Relativistic momentum of a point particle.

The Hamiltonian is given by:

$$H(\vec{x},\vec{\Pi}) = \dot{\vec{x}}\cdot\vec{\Pi} - L(\vec{x}, \dot{\vec{x}}(\vec{x},\vec{\Pi}))$$

Lagrangian → Hamiltonian: go from variable $(\vec{x},\dot{\vec{x}})$ coordinates and velocities, to $(\vec{x},\vec{\Pi})$ coordinates & canonical momenta by Legendre transformation.

For $L = -mc^2\sqrt{1-\vec{v}^2/c^2}$ $\quad \vec{\Pi} = \dfrac{m\vec{v}}{\sqrt{1-\vec{v}^2/c^2}}$.

$$H = \vec{\Pi}\cdot\vec{v} - L = \frac{mv^2}{\sqrt{1-v^2/c^2}} + mc^2\sqrt{1-v^2/c^2} = \frac{mc^2}{\sqrt{1-v^2/c^2}}$$

The answer coincides with the relativistic energy of the point particle.

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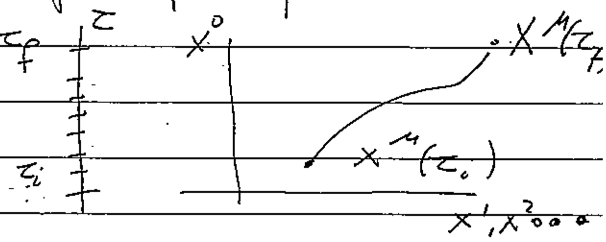
Reparameterization invariance

As we will see later on an important property of string theory is reparameterization invariance. we start with the example of the point particle.

we parametrize the world-line of the point particle with a parameter τ.



$$X^\mu = X^\mu(\tau) \quad \& \quad X_i^\mu = X^\mu(\tau_i) \quad X_f^\mu = X^\mu(\tau_f).$$

we can write the action in terms of τ.

in terms of τ $\quad ds = \eta_{\mu\nu}\, dX^\mu\, dX^\nu = \eta_{\mu\nu} \dfrac{dX^\mu}{d\tau}\dfrac{dX^\nu}{d\tau}(d\tau)$

$$\Rightarrow S' = -mc \int_{\tau_i}^{\tau_f} \sqrt{\eta_{\mu\nu}\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}}\, d\tau.$$

The action is invariant under reparameterization.

suppose we take $\tau \to \tau' = \tau'(\tau)$

then $\dfrac{dX^\mu}{d\tau} = \dfrac{dX^\mu}{d\tau'}\dfrac{d\tau'}{d\tau}.$

substituting into the action we get.

$$S' = -mc \int_{\tau_i}^{\tau_f} \sqrt{\eta_{\mu\nu}\frac{dX^\mu}{d\tau'}\frac{dX^\nu}{d\tau'}} \cdot \frac{d\tau'}{d\tau}\, d\tau = -mc \int_{\tau_i'}^{\tau_f'} \sqrt{\eta_{\mu\nu}\frac{dX^\mu}{d\tau'}\frac{dX^\nu}{d\tau'}}\, d\tau'$$

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The action is invariant under:

* Reparameterizations $\tau \to \tau'(\tau)$

* Poincare transformations: $X^\mu \to X'^\mu = \Lambda^\mu{}_\nu X^\nu + a^\mu$

where $\Lambda^\mu{}_\nu$ are Lorentz transformations and a^μ are constants

Poincare invariance is manifest because the action is a Lorentz scalar and is constructed of Lorentz vectors

Equations of motion

As usual we obtain the E.o.M from the variation $\delta_1 S' = 0$ by varying the path $X^\mu(\tau) \to X^\mu(\tau) + \delta X^\mu(\tau)$

$$\delta_1 S' = -mc \int \delta(ds)$$

we can find $\delta(ds)$ by taking $\delta(ds)^2 = \delta(\eta_{\mu\nu}\, dx^\mu dx^\nu)$

$$\Rightarrow 2\, ds\, \delta(ds) = 2\eta_{\mu\nu}\, \delta(dX^\mu)\, dX^\nu \leftarrow (\text{from symmetry of } \eta_{\mu\nu} dx^\mu dx^\nu)$$

$$\Rightarrow \delta(ds) = \eta_{\mu\nu}\, \delta(dX^\mu) \frac{dX^\nu}{ds}$$

use $\delta(dX^\mu) = d(\delta X^\mu)$ (as $d(a(\tau)) = a(\tau+d\tau) - a(\tau)$ and δ is linear)

hence: $\delta(ds) = \eta_{\mu\nu}\, d\,\delta X^\mu \dfrac{dX^\nu}{ds} =$

$$\eta_{\mu\nu} \frac{d\delta X^\mu}{d\tau} \frac{dX^\nu}{ds}\, d\tau.$$

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$$\Rightarrow \delta_1 S' = -mc \int \delta(ds) = -mc \int_{\tau_i}^{\tau_f} \eta_{\mu\nu} \frac{d\,\delta(X^\mu)}{d\tau} \frac{dX^\nu}{ds} d\tau.$$

I.b.P. $\Rightarrow$

$$= -mc \int_{\tau_i}^{\tau_f} d\tau \frac{d}{d\tau}\left(\eta_{\mu\nu}\,\delta(X^\mu(\tau))\frac{dX^\nu}{ds}\right) + \int_{\tau_i}^{\tau_f} d\tau\, \delta(X^\mu(\tau))\left(mc\,\eta_{\mu\nu}\frac{d}{d\tau}\left(\frac{dX^\nu}{ds}\right)\right)$$

we set $\delta(X^\mu(\tau_i)) = \delta(X^\mu(\tau_f)) = 0 \Rightarrow$ first term vanishes.

$$\Rightarrow \delta_1 S' = \int_{\tau_i}^{\tau_f} d\tau\, \delta(X^\mu(\tau))\, mc\, \eta_{\mu\nu} \frac{d}{d\tau}\left(\frac{dX^\nu}{ds}\right)$$

Recall that: $mc \dfrac{dX^\nu}{ds} = m u^\nu = P^\nu$

$$\Rightarrow \delta_1 S' = \int_{\tau_i}^{\tau_f} d\tau\, \delta(X^\mu(\tau))\, \eta_{\mu\nu} \frac{dP^\nu}{d\tau} = \int_{\tau_i}^{\tau_f} d\tau\, \delta(X^\mu(\tau)) \frac{dP_\mu}{d\tau}.$$

since $\delta(X^\mu(\tau))$ is arbitrary

$$\Rightarrow \delta_1 S' = 0 \Rightarrow \frac{dP_\mu}{d\tau} = 0.$$

$\Rightarrow$ the momentum of a point particle is constant along its world-line. τ is an arbitrary parameter that parameterizes the world-line

$$\Rightarrow \frac{dP^\mu}{d\tau} = \frac{dP^\mu}{d\tau'}\left(\frac{d\tau'}{d\tau}\right) = 0 \Rightarrow \frac{dP^\mu}{d\tau'} = 0.$$

where $\tau' = \tau'(\tau)$ and $\dfrac{d\tau'}{d\tau} \neq 0$.

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using proper time $\Rightarrow \frac{dp^\mu}{ds} = 0 \Rightarrow m\frac{d^2 x^\mu}{ds^2} = 0.$

This equation of motion is invariant under Lorentz transformations

take $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \qquad \Lambda^\mu{}_\nu = \text{constants}$

$$0 = \frac{d^2 x'^\mu}{ds^2} = \frac{d^2}{ds^2}\Lambda^\mu{}_\nu x^\nu = \Lambda^\mu{}_\nu \frac{d^2 x^\nu}{ds^2} \Rightarrow \frac{d^2 x^\nu}{ds^2} = 0$$

Relativistic Particle with external force

so far we discussed a free point particle.

we can generalise this for a particle subject to an external force,

The relativistic version of Newton's Eq:

$$(*) \qquad \frac{d\vec{p}}{dt} = \frac{d}{dt}\left(\frac{m\vec{v}}{\sqrt{1-v^2/c^2}}\right) = \vec{F}$$

OR in covariant form. (set $c = 1$)

$$\frac{d}{ds}\left(m\frac{dx^\mu}{ds}\right) = f^\mu \qquad s\text{ - proper time.}$$

where $$f^\mu = \left(\frac{\vec{v}\cdot\vec{F}, \vec{F}}{\sqrt{1-v^2}}\right).$$

we want to see that this is the same as $(*)$

Note $\vec{v}\cdot\frac{d\vec{p}}{dt} = \vec{v}\frac{d}{dt}\left(\frac{m\vec{v}}{(1-v^2)^{1/2}}\right) = \vec{v}\cdot\frac{m\frac{d\vec{v}}{dt}}{(1-v^2)^{3/2}} = \frac{d}{dt}\left(\frac{m}{(1-v^2)^{1/2}}\right) = \frac{d}{d}$

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$\Rightarrow \quad \dfrac{dp^0}{dt} = \vec{v} \cdot \vec{F}$.

$\Rightarrow \quad \dfrac{dp^0}{ds}\dfrac{ds}{dt} = \vec{v} \cdot \vec{F}$ OR $\dfrac{dp^0}{ds} = \dfrac{\vec{v} \cdot \vec{F}}{(1-v^2)^{1/2}}$

$\Rightarrow \quad \dfrac{d}{ds}\left(p^\mu\right) = f^\mu$ is the covariant form.

$\longrightarrow$ we want to write down the action producing this eq

* if the force f^μ comes from a potential

$$f^\mu = \partial^\mu V(x)$$

$$\Rightarrow S' = -m \int_{\tau_i}^{\tau_f} \sqrt{\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}\, d\tau - \int_{W.\,line} V(x(\tau))\, d\tau.$$

* if f^μ is the Lorentz force acting on a particle with charge $q \Rightarrow f^\mu = q F^{\mu\nu} \dfrac{dx_\nu}{ds}$

$$\Rightarrow S = -m \int \sqrt{\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}\, d\tau - q \int A_\mu\, dx^\mu$$

$$= -m \int \sqrt{\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}\, d\tau - q \int A_\mu \frac{dx^\mu}{d\tau}\, d\tau.$$

This gives the eq. of motion.

$$\frac{d}{d\tau}\left(m \frac{dx^\mu}{d\tau}\right) = q F^{\mu\nu} \frac{dx_\nu}{d\tau}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \longrightarrow$ field strength tensor.

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This is the covariant form of:

$$\frac{d\vec{P}}{dt} = q\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

Canonical momenta and Hamiltonian for the covariant action.

$\boxed{c=1}$ action and Lagrangian $\boxed{c=1}$

$$S = \int L\,dt = -m\int\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}\,d\tau$$

canonical momenta

check signs

$$\pi^\mu = \frac{\partial L}{\partial\left(\frac{dx^\mu}{d\tau}\right)} = +m\frac{\frac{dx^\mu}{d\tau}}{\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}} = +m\frac{dx}{ds}$$

$$\pi_\mu\pi^\mu = -m^2$$

For proper time $\pi^\mu = p^\mu \rightarrow p_\mu p^\mu = -m^2$

Hamiltonian | H

$$H = \pi^\mu\frac{dx_\mu}{d\tau} - L = -\frac{m\frac{dx^\mu}{d\tau}}{\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}}\frac{dx_\mu}{d\tau} + m\sqrt{\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} = 0$$

$\Rightarrow$ Hamiltonian $\neq$ total energy Rather $H=0$

This reflect the fact that the momenta are not independent. $P^2 = m^2$

Write $H = \lambda(P^2 - m^2)$

(typical for covariant momenta).

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Covariant action for massless and massless particles

we need to include massless particles.

Introduce: Invariant line element: $e(\tau)d\tau$ such that,

$$\left.\begin{aligned}\tilde{e}(\tilde{\tau}) &= e(\tau)\frac{d\tau}{d\tilde{\tau}} \\ d\tilde{\tau} &= \frac{d\tilde{\tau}}{d\tau}d\tau\end{aligned}\right\} \Rightarrow \tilde{e}(\tilde{\tau})d\tilde{\tau} = e(\tau)\frac{d\tau}{d\tilde{\tau}}\frac{d\tilde{\tau}}{d\tau}d\tau = e(\tau)d\tau$$

$\rightarrow$ $e(\tau)d\tau$ is invariant under reparameterizations.

Action $$S[X,e] = \frac{1}{2}\int d\tau\left(\frac{1}{e}\left(\frac{dX^\mu}{d\tau}\right)^2 - m^2 e\right)$$

Symmetries of the action.

* $S[X,e]$ is invariant under $\tau \to \tilde{\tau}$ (Reparameterizations)
* $S[X,e]$ is invariant under $X^\mu \to \Lambda^\mu{}_\nu X^\nu + a^\mu$ (Poincare)

The action depends on the fields X^μ and e.

The corresponding eq. of motion are found by varying $X \to X + \delta X$ and $e \to e + \delta e$.

We obtain: The Eqs. of motion from the Euler-Lagrange Eqs.

e: $$\frac{d}{d\tau}\frac{\partial L}{\partial\left(\frac{\partial e}{\partial \tau}\right)} - \frac{\partial L}{\partial e} = -\frac{\partial L}{\partial e} = -\frac{1}{2}\left(-\frac{1}{e^2}\left(\frac{dX^\mu}{d\tau}\right)^2 - m^2\right) = 0$$

$$\Rightarrow \left(\frac{dX^\mu}{d\tau}\right)^2 + e^2 m^2 = 0$$

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X^μ: $\frac{d}{d\tau}\frac{\partial L}{\partial\left(\frac{\partial X^\mu}{\partial\tau}\right)} \quad \frac{\partial L}{\partial X^\mu} = \frac{d}{d\tau}\left(\frac{1}{e}\frac{dX^\mu}{d\tau}\right) = 0$

The equation of motion for e is algebraic.

e is an auxiliary parameter, not a dynamical field.

if $m^2 \neq 0$, we can solve for e.

$$e^2 = -\frac{1}{m^2}\left(\frac{dX^\mu}{d\tau}\right)^2$$

For massive paricles $m^2 > 0$ $\left(\frac{dX^\mu}{d\tau}\right)^2 < 0$.

Recall: $\left(\frac{dX^\mu}{d\tau}\right)^2 = \eta_{\mu\nu}\left(\frac{dX^\mu}{d\tau}\right)\left(\frac{dX^\nu}{d\tau}\right) = -c^2\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{d\vec{X}}{d\tau}\right)^2 < 0$

For time-like distances.

hence $e^2 > 0 \Rightarrow e = \frac{\sqrt{-(dX^\mu/d\tau)^2}}{m}$

substituting this solution back into the action

we recover: $S = \frac{1}{2}\int d\tau\left((-m)\cdot\frac{(-X'^2)}{\sqrt{-X'^2}} - \sqrt{-X'^2}\cdot m\right) =$

$\boxed{X' = \frac{dX^\mu}{d\tau}}$

$$= -m\int d\tau\sqrt{-\eta_{\mu\nu}\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}}$$

which is the same as the action that we saw before

for a massive particle.

However, the new action allows us to deal with massless Particles as well.

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according to the equation: $X'^2 + e^2 m^2 = 0$

e controls the norm of the tangent vector to the world-line that we can change by reparameterization.

Instead of eliminating e by its equation of motion we can fix it in a prescribed 'gauge' (= parameterization)

* For $m^2 > 0$, impose the gauge.

$$e = \frac{1}{m}.$$

The equations of motion become:

$$\frac{d}{d\tau}\frac{dX^\mu}{d\tau} = \frac{d^2 X^\mu}{d\tau^2} = 0.$$

$$X'^2 = \eta_{\mu\nu}\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau} = -1$$

Comment: in this gauge the parameter τ is the proper time.

* For $m^2 = 0$, impose the gauge

$$e = 1$$

The equations of motion become

$$\frac{d^2}{d\tau^2} X^\mu = 0 \qquad \text{(as before)}$$

$$X'^2 = 0.$$

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Comment: For $m^2=0$ the eq. $x'^2+m^2=0$ tells us that the world-line is light-like, i.e.

$$-c^2dt^2+dx^2+dy^2+dz^2=0.$$

as expected for a massless particle.

In this case there is no proper time, but one can choose a so-called affine parameterization. $e=1$ (or any other constant value) corresponds to picking an affine parameter.

Note | In both cases the dynamical equation of motion

$$\frac{d^2x^\mu}{d\tau^2}=0$$

must be supplemented by a constraint

$$\phi=\left\{\frac{\mathring{x}^2}{\mathring{x}^2}+1\right\}=0.$$

to capture the full information.