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In Extra dimensions.

$$[X^i, P_j] = i\hbar \delta^i{}_j$$

where

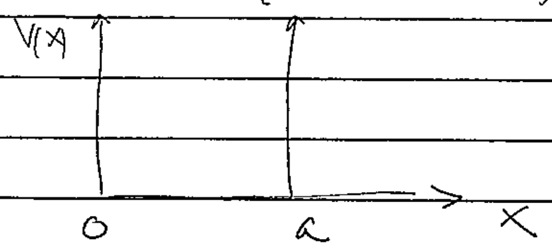
$$\delta^i{}_j = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

consider the time-independent schrödinger eq. in 1-dimension.

$$-\frac{\hbar^2}{2m}\vec{\nabla}^2 \psi(x) + V(x)\psi(x) = E\psi(x)$$

in the case of of a square-well potential of infinite height.

$$V(x) = \begin{cases} 0 & x \in (0,a) \\ \infty & x \notin (0,a) \end{cases}$$



The boundary conditions imply that

$$\psi(x) = 0 \quad \text{for} \quad x \notin (0,a)$$

continuity of the wave function on the boundary implies that $\psi(0) = \psi(a) = 0$.

$\Rightarrow$ quantum mechanics of a particle confined to the segment $0 \leq x \leq a$.

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For $0 \leq x \leq a$. $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi = E\psi$ (*)

$\Rightarrow \psi(x) = A\sin\lambda x + B\cos\lambda x.$

$\psi(0) = 0 \Rightarrow B = 0.$

$\psi(a) = 0 \Rightarrow A\sin\lambda a = 0 \Rightarrow \lambda = \frac{k\pi}{a}$

$k = 1, 2, 3 \dots \infty$

sub into (*) $\Rightarrow +\frac{\hbar^2}{2m}\frac{k^2\pi^2}{a^2} A\sin\frac{k\pi}{a}.x = EA\sin\frac{k\pi}{a}x$

$\Rightarrow E = \frac{\hbar^2 k^2 \pi^2}{2ma^2}$

To find A impose $\int_{-\omega}^{\omega}|\psi|^2\,dx = 1 = \int_{-\omega}^{\omega}\rho\,dx$

$\cos^2-\sin^2=\cos 2x$
$\cos^2+\sin^2=1$

$\int_0^a A^2\sin^2\frac{k\pi}{a}x\,dx = A^2\int_0^a \frac{1}{2}\left(1-\cos\frac{2k\pi x}{a}\right)dx =$

$= \frac{A^2}{2}\left(x\Big|_0^a - \frac{a}{2k\pi}\sin 2\frac{k\pi}{a}x\Big|_0^a\right) = \frac{A^2 a}{2} = 1$

$\Rightarrow A = \sqrt{\frac{2}{a}} \Rightarrow \psi(x) = \sqrt{\frac{2}{a}}\sin\frac{k\pi x}{a}.$

square well with EXTRA dimension.

Consider the case with an extra compact dimension.

$(x, y) \sim (x, y+2\pi R).$

with

$$V(x,y) = \begin{cases}\infty & x \notin (0,a)\\ 0 & x \in (0,a)\end{cases}$$

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we will consider the case with $R \ll a$.

The Schrödinger eq. in two dimensions is:

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right) = E\psi$$

using separation of variables.

$$\psi(x,y) = G(x)\,\phi(y)$$

$$\Rightarrow -\frac{\hbar^2}{2m}\left(G_{xx}\phi + G\,\phi_{yy}\right) = E\,G(x)\phi(y)$$

$$\Rightarrow -\frac{\hbar^2}{2m}\left(\frac{G_{xx}}{G}+\frac{\phi_{yy}}{\phi}\right) = E = \text{const.}$$

$$\Rightarrow -\frac{\hbar^2}{2m}\frac{G_{xx}}{G} = C_1 \qquad -\frac{\hbar^2}{2m}\frac{\phi_{yy}}{\phi} = C_2 \quad \text{with } C_1+C_2=E$$

we solve the two equations subject to the boundary conditions with $\phi(y) = \phi(y+2\pi R)$.

in x- direction we have as before

$$G_k(x) = C_k \sin\frac{k\pi x}{a}$$

in y-direction |

$$\phi_l(y) = a_l \sin\frac{l\,y}{R} + b_l \cos\frac{l\,y}{R}.$$

for $l=0$ we have

$$\phi_0(y) = b_0$$

Putting back the solutions into the equations for x and y

$$\Rightarrow E_{k,l} = C_1 + C_2 = \frac{\hbar^2}{2m}\left[\frac{k^2\pi^2}{a^2} + \frac{l^2}{R^2}\right]$$

new : For $l \neq 0$ the spectrum is doubly degenerate $\pm l$.

For $l = 0$ the energy states are the same as the 1-dimensional case.

Lowest new state

The lowest new energy level is for $l=1$ $k=1$.

$$E_{1,1} = \frac{\hbar^2}{2m}\left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{1}{R}\right)^2\right]$$

in the case with $R << a$

$$E_{1,1} \sim \frac{\hbar^2}{2m}\frac{1}{R^2}$$

This energy is comparable to the the k-level eigenstates of the one dimensional problem when

$$\frac{k\pi}{a} \sim \frac{1}{R} \longrightarrow k \sim \frac{1}{\pi}\frac{a}{R}.$$

Since $R << a$ k is very large.

$\Rightarrow$ The first new energy levels are at an energy far above those of the low-lying original states.

$\rightarrow$ The extra dimension remains hidden provided that $R << a$

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As we will see later strings introduce a new feature.

$$E \sim \frac{m^2}{R^2} + \frac{n^2 R^2}{\alpha'^2}$$ ⓞ

strings can wrap around the compact dimension.

and create new states ~ wrapping modes.

For point particle → no wrapping modes $E \sim \frac{m^2}{R^2}$.

Elecromagnetism and gravitation in various dimensions

Classical Electrodynamics | Maxwell's equation describe the dynamics of electric and magnetic fields in the presence of electric charges and currents.

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial B}{\partial t}$$

$$\vec{\nabla} \cdot B = 0$$

$$\vec{\nabla} \cdot E = \rho$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

$\vec{E}, \vec{B}$ are the three dimensional electric and magnetic fields

ρ, J are the electric charge and current densities

Lorentz Law: $\frac{d\vec{P}}{dt} = q\left(\vec{E} + \frac{\vec{U}}{c} \times \vec{B}\right)$ → effect of E & B fields on a moving charged

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since $\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla}\times\vec{A}$

because $(\vec{\nabla}\cdot(\vec{\nabla}\times A)) = \partial_i \epsilon_{ijk}\partial_j A_k = \epsilon_{ijk}\partial_i\partial_j A_k =$

$= -\epsilon_{jik}\partial_i\partial_j A_k = -\epsilon_{ijk}\partial_j\partial_i A_k =$

$= -\epsilon_{ijk}\partial_i\partial_j A_k \Rightarrow \epsilon_{ijk}\partial_i\partial_j A_k = 0.$

using our summation convention and : $\epsilon_{ijk} = \begin{cases} +1 & \epsilon_{123} \\ -1 & \epsilon_{213} \\ 0 & \epsilon_{113} \end{cases}$ etc.. $i,j,k = 1,2,3.$

in electrostatics. $\vec{\nabla}\times\vec{E} = 0 \Rightarrow \vec{E} = -\vec{\nabla}\phi$

in electrodynamics $\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\times\vec{A}$ (FROM Maxwell eq.)

$$\Rightarrow \vec{\nabla}\times\left(\underline{E} + \frac{1}{c}\frac{\partial A}{\partial t}\right) = 0$$

$$\Rightarrow \vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}.$$

$\Rightarrow$ we can describe E & M dynamics in terms of the potential functions $\vec{A}$ & ϕ.

<u>Manifestly relativistic electrodynamics</u>

$A^{\mu} = (\Phi, A^1, A^2, A^3) \rightarrow$ 4-vector.

$A_{\mu} = (-\Phi, +\vec{A})$

From A_{μ} we create the electomagnetic field strength

$$F_{\mu\nu} = (\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}).$$

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we have

$$\vec{F}_{\nu\mu} = \partial_\nu A_\mu - \partial_\mu A_\nu = -(\partial_\mu A_\nu - \partial_\nu A_\mu) = -\vec{F}_{\mu\nu}.$$

$\Rightarrow$ $F_{\mu\nu}$ is antisymmetric.

$\Rightarrow$ diagonal components of $F_{\mu\nu}$ vanish.

let i denote a spatial index i.e. $i = 1, 2, 3$.

Then $F_{0i} = \partial_0(+A_i) - \partial_i A_0 = \frac{1}{c}\frac{\partial}{\partial t}(+A^i) + \frac{\partial}{\partial X^i}\Phi = -E_i$

$$F_{ij} = \partial_i(+A_j) - \partial_j(+A_i) = +(\partial_i A_j - \partial_j A_i) = +(\vec{\nabla}\times\vec{A})_k = +B_k$$

Hence

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ +E_1 & 0 & +B_3 & -B_2 \\ +E_2 & -B_3 & 0 & +B_1 \\ +E_3 & +B_2 & -B_1 & 0 \end{pmatrix},$$

$\rightarrow$ The electric and magnetic fields are encoded in The field stregth tensor $F_{\mu\nu}$

gauge transformations: The potential A^μ can be changed by gauge transformations

$$A_\mu \longrightarrow A'_\mu = A_\mu + \partial_\mu \epsilon(X^\mu)$$

where $\epsilon(X^\mu)$ is a scalar function of X^μ.

without affecting $F_{\mu\nu}$

$F_{\mu\nu} \longrightarrow F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = \partial_\mu(A_\nu + \partial_\nu\epsilon) - \partial_\nu(A_\mu + \partial_\mu\epsilon) = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}$

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In terms of ϕ, $\vec{A}$ we have.

$$\phi \to \phi' = \phi - \frac{1}{c}\frac{\partial \epsilon}{\partial t}$$
$$\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\epsilon,$$

and $\vec{E}$ and $\vec{B}$ are unchanged.

i.e. $$\vec{E}' = -\vec{\nabla}\phi' - \frac{1}{c}\frac{\partial \vec{A}'}{\partial t} = -\vec{\nabla}\left(\phi - \frac{1}{c}\frac{\partial \epsilon}{\partial t}\right) - \frac{1}{c}\frac{\partial}{\partial t}\left(A + \vec{\nabla}\epsilon\right)$$
$$= -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial}{\partial t}\vec{A} + \cancel{\frac{1}{c}\vec{\nabla}\frac{\partial}{\partial t}\epsilon} - \cancel{\frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\epsilon} = \vec{E},$$

and $$\vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\left(\vec{A} + \vec{\nabla}\epsilon\right) = \vec{\nabla}\times\vec{A} + \vec{\nabla}\times\vec{\nabla}\epsilon = \vec{\nabla}\times\vec{A} = \vec{B},$$

(because $\vec{\nabla}\times\vec{\nabla}\epsilon = \epsilon_{ijk}\partial_j\partial_k\epsilon = -\epsilon_{ikj}\partial_j\partial_k\epsilon = -\epsilon_{ikj}\partial_k\partial_j\epsilon = -\vec{\nabla}\times\vec{\nabla}\epsilon$)

Next we write Maxwell's equations in a manifestly Relativistic form.

consider | $T_{\lambda\mu\nu} = \partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda}$

$T_{\lambda\mu\nu}$ is identically zero

because | $T_{\lambda\mu\nu} = \partial_\lambda(\partial_\mu A_\nu - \partial_\nu A_\mu) + \partial_\nu(\partial_\lambda A_\mu - \partial_\mu A_\lambda) + \partial_\mu(\partial_\nu A_\lambda - \partial_\lambda A_\nu) \equiv 0$

hence | $\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0.$

using $F_{\mu\nu} = -F_{\nu\mu}$ it follows that,

$$T_{\lambda\mu\nu} = -T_{\lambda\nu\mu} \qquad T_{\lambda\mu\nu} = -T_{\mu\lambda\nu}$$

i.e. $T_{\lambda\mu\nu}$ is antisymmetric under exchange of any two indecies

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The source force Maxwell equations are obtained

from. $T_{\lambda\mu\nu} = \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} \equiv 0$

Since $T_{\lambda\mu\nu}$ is totally antisymmetric and $T_{\lambda\mu\nu} \equiv 0$.

$\Rightarrow$ 4 independent equations.

e.g. $\lambda = 0 \quad \mu = i \quad \nu = j$.

$$T_{0ij} = \partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} =$$

$$= +\partial_t(+B_k) + \partial_i(+E_j) - \partial_j E_i =$$

$$= +\partial_t B_k + \epsilon_{kij}\partial_i E_j = 0$$

$$\Rightarrow (\nabla \times E)_k + \frac{\partial(\vec{B})_k}{\partial t} = 0.$$

FOR $\lambda = i \quad \mu = j \quad \nu = k$.

$$T_{ijk} = \partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij}$$

take $i=1$ $j=2$ $k=3$ $\Rightarrow \partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12}$

$$= \partial_1(+B_1) + \partial_2(+B_2) + \partial_3(+B_3) = 0$$

$$\Rightarrow (\vec{\nabla} \cdot \vec{B}) = 0$$

The two Remaining Maxwell equations

contain electric charges and currents.

introduce: $j^\mu = (c\rho, j^1, j^2, j^3) \rightarrow$ FOUR vector current density

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$\rho \rightarrow$ charge density.

$\vec{J} = (j_1, j_2, j_3)$ current density.

introduce | $F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta} F_{\alpha\beta}$

Show (1) $F^{\mu\nu} = -F^{\nu\mu}$ $\quad F^{0i} = -F_{0i}$. $\quad F^{ij} = F_{ij}$.

2) $F^{\mu\nu} = (\partial^\mu A^\nu - \partial^\nu A^\mu)$

3) $$F^{\mu\nu} = \begin{pmatrix} 0 & +E_1 & +E_2 & +E_3 \\ -E_1 & 0 & +B_3 & -B_2 \\ -E_2 & -B_3 & 0 & +B_1 \\ -E_3 & +B_2 & -B_1 & 0 \end{pmatrix}$$

The remaining Maxwell equations the take the form.

$$\partial_\nu F^{\mu\nu} = \frac{\partial}{\partial x^\nu} F^{\mu\nu} = \frac{1}{c} j^\mu \qquad \text{(HW. 2)}$$

In the absence of sources.

$$\Rightarrow \partial_\nu F^{\mu\nu} = 0 \rightarrow \partial_\nu \partial^\mu A^\nu - \partial_\nu \partial^\nu A^\mu = 0.$$

$$\text{or } \partial^2 A^\mu - \partial^\mu (\partial \cdot A) = 0$$

where $\partial_\nu \partial^\nu \equiv \partial^2$.

we can now use these equations to generalise.

Maxwell equations to any dimensions

- 4D $\quad 0, 1, 2, 3. \qquad \eta^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}$

nD $\quad 0, 1, 2, \ldots, n \qquad \eta^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & +1 & & \\ & & +1 & \\ & & & \ddots \end{pmatrix}_{n \times n}$

Example: 3D space-time

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y \\ +E_x & 0 & +B_z \\ +E_y & -B_z & 0 \end{pmatrix}.$$

in D dimension: $F^{0i} = -F_{0i} = +E_i \Rightarrow$ electric field.

$F_{ij} = -F_{ji} \Rightarrow$ magnetic field

in D-dimension E_i is a spatial vector $i=1,..D$

B_i is not a spatial vector.

spheres in higher dimensions.

a 2-sphere in 3 Dimensions defined by the equation,

$$S^2(R) = X_1^2 + X_2^2 + X_3^2 = R^2$$

The volume enclosed by the sphere is a ball.

$$B^3(R) = X_1^2 + X_2^2 + X_3^2 \leq R^2$$

where R is the Radius of the ball.

similarly in D-dimensions.

$$S^{D-1}(R) = X_1^2 + X_2^2 + \cdots + X_D^2 = R^2.$$

$$B^D(R) = X_1^2 + X_2^2 + \cdots + X_D^2 \leq R^2.$$

the volume of a 1-spheres: $vol(S^1(R)) = 2\pi R.$

" " " a 2-sphere: $vol(S^2(R)) = 4\pi R^2$

" " " a D-1 sphere. $vol(S^{D-1}(R)) = vol(S^{D-1})R^{D-1}$