LLST.1 | 1/2/16.1 | Liverpool. Lect.1 | why string ?

physics: Mathematical formulation of experimental observations

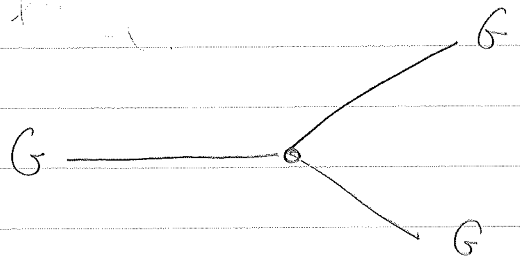
unification: encompassing mathematical model.

strings: unification of:

Gravity	Gauge & Matter.
cellestial & cosmological	subatomic particle & fields
Riemanian Geometry	Quantum field theory.
General relativity	Standard Model.

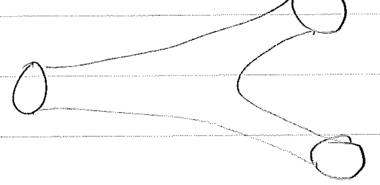
yet.

General relativity & QFT's are incompatible at a fundamental level.



$\longrightarrow \infty$.

strings



$\longrightarrow$ finite.

L1 ST. 2 | 2/2/16.1 | Liverpool |

Inventory.

Forces | E.M. × Weak   Strong   spin +1 particles

symmetries. $U(1) \times SU(2)$ $SU(3)$ local phase invariance.

Gravity.   spin 2 particle.

Matter: QUARKS & Leptons.

QUARKS ↓ strong, weak, E.M.

Leptons: weak, E.M.

quarks: $\begin{pmatrix} u \\ \text{down} \end{pmatrix}$ $\begin{pmatrix} c \\ s \end{pmatrix}$ $\begin{pmatrix} t \\ b \end{pmatrix}$

leptons: $\begin{pmatrix} \nu_e \\ e \end{pmatrix}$ $\begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}$ $\begin{pmatrix} \nu_e \\ \tau \end{pmatrix}$.

mass → spontaneous symmetry breaking → Higgs

$SU(2) \times U(1) \rightarrow U(1)_{e.m.}$

Unification: GUTs

$SU(3) \times SU(2) \times U(1) \rightarrow SU(5) \rightarrow SO(10)$

SO → simple orthogonal 10×10 matrices : generators $O^T = -O$

$$\frac{n^2-n}{2} + n = \frac{n(n+1)}{2} \longrightarrow \text{symmetric.}$$

$$\frac{n^2-n}{2} = \frac{n(n-1)}{2} \longrightarrow \text{asymmetric } (i)$$

hermitian $O^+ = O$

LLST, 3 | 27/1/14, | Liverpool | (Revised).

Time evolution: $|a(0)\rangle \to |a(t)\rangle$

$$|a(t)\rangle = U\, |a(0)\rangle$$

Physical theories $\to$ Unitary $U^{+}U = 1$

$$\Rightarrow \langle a(t)|a(t)\rangle = \langle a(0)|U^{+}U|a(0)\rangle = \langle a(0)|a(0)\rangle$$

Representation of a unitary operator:

$$U = e^{iH} \leftarrow \text{hermitian}$$

$H \to$ generators of, a Lie algebra

e.g. $SU(n)$, $SO(2n)$.

$$\overset{n=5}{SO(10)} \to \overset{2}{SU(3)} \times \overset{1}{SU(2)} \times \overset{1}{U(1)_x} \times \overset{1}{U(1)_y}$$

$$45 = (8,1,0,0) + (1,3,0,0)\ (3,2,x,y) + (\bar{3},2,-x,-y)$$

$$+ (6,1,x',y') + (\bar{6},1,-x',-y')$$

$$+ (1,3,x'',y'') + (1,3,-x'',-y'')$$

$$+ (1,1,x''',y''') + (1,1,-x''',-y''').$$

LLST.3' | 27/1/14.2 | Liverpool / Revised |

FOR OUR purpose: chiral 16 representation of SO(10) put even − in 5 slots

3			2		
+	+	+	+	+	e_L^c
+	+	+	−	−	N_L^c
+	+	−	+	−	
+	−	+	+	−	
−	+	+	+	−	$Q = \begin{pmatrix} u \\ d \end{pmatrix}$
+	+	−	−	+	
+	−	+	−	+	
−	+	+	−	+	
−	−	+	+	+	
−	+	−	+	+	D_L^c
+	−	−	+	+	
−	−	−	−	+	L
−	−	−	+	−	
−	−	+	−	−	
−	+	−	−	−	U_L^c
+	−	−	−	−	

LLST.4 | 26/6/09.4 | Abingdon.

OR using $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$$16 = \binom{3}{0}\binom{2}{0} + \binom{3}{0}\binom{2}{2} + \binom{3}{1}\binom{2}{1} + \binom{3}{2}\binom{2}{0} + \binom{3}{2}\binom{2}{2} + \binom{3}{3}\binom{2}{1}$$

$(1,1)$ .. $(1,1)$ .. $(3,2)$ .. $(\bar{3},1)$ .. $(\bar{3},1)$ .. $(1,2)$ ..

E_L^c N_L^c Q_L D_L^c U_L^c L_L.

These are the sixteen states in a one S. M. generation.

+ R. H. N.

There is something right about GUTs (gauged).

Problems | Hierachy ? $M_W \sim 100$ GeV, $M_{GUT} \sim 10^{16}$ GeV ?

Flavour ? Not one but three ?

GUTs: gauge theories → renormalizable local quantum field theories

Renormalizable : infinities absorbed in a finite number of measurable parameters. →

Gravity → nonrenormalizable.

→ Quantum & Gravity ?

→ String → finite → Gravity & Gauge unification.

LLST.5 / 3/7/09.1 / Abingdon / Friday /

special relativity and extra dimensions

Einstein's special relativity: 1) C = speed of light = constant in all inertial frames

$c = 3 \times 10^8 \frac{m}{s}$.

2) Laws of physics are the same in all inertial frames.

events are labeled by their time & position in

Inertial frames. $\rightarrow$ 4-vector: $X^\mu = (ct, \vec{X})$

$X^0 = ct \qquad \vec{X} = (X^1, X^2, X^3) = (X, y, z) \qquad \mu = 0, 1, 2, 3$

The length of four vector is given by:

$$X \cdot X = -c^2t^2 + X^2 + y^2 + z^2$$

$$= \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu} X^\mu X^\nu = X_\mu X^\mu$$

Einstein summation convention.

where $\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}$ — 1 time, 3 space

is the Minkowski metric.

In general the scalar product of two four vectors

$X^\mu; y^\nu$ is given by $X \cdot y = \sum_{\mu\nu=0}^{3} \eta_{\mu\nu} X^\mu y^\nu = X_\nu y^\nu$

$\boxed{X_\nu = \eta_{\mu\nu} X^\mu}$

Lorentz transformations:

From one inertial frame X^μ

To another " " X'^μ.

LLST. 6 | 3/7/09. 2 | Abingdon | Friday |

The Lorentz transformations preserve the scalar product.

$$-c^2t^2 + \vec{x}^2 = X \cdot X = \eta_{\mu\nu} X^\mu X^\nu = X' \cdot X' = \eta_{\mu\nu} X'^\mu X'^\nu = -c^2 t'^2 + \vec{x}'^2$$

where X and X' are related by a Lorentz transformation.

$$X^\mu \rightarrow X'^\mu = \Lambda^\mu{}_\nu X^\nu$$

Think for simplicity with the more familiar case of ordinary rotations in 3 dimensions.

$$\vec{X} \rightarrow \vec{y} = R\vec{X}.$$

Rotations preserve the length of a three vector.

Similarly; Lorentz transformations preserve the length of a four vector in 4 dimensional Minkowski space-time.

*** | **Properties of Lorentz transformations**

$\eta_{\mu\nu} X^\mu X^\nu \rightarrow$ invariant under L.T.

No change in size or shape

$$\eta_{\mu\nu} X^\mu X^\nu \longrightarrow \eta_{\mu\nu} X'^\mu X'^\nu = \eta_{\mu\nu} \Lambda^\mu{}_\alpha X^\alpha \Lambda^\nu{}_\beta X^\beta$$

$$= \eta_{\alpha\beta} X^\alpha X^\beta$$

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$\Rightarrow (*) \quad \eta_{\mu\nu} \Lambda^{\mu}{}_{\nu} \Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta}$

In matrix notation $\quad \Lambda^T \eta \Lambda = \eta$.

$\rightarrow$ Defines the Lorentz transformations.

$\Rightarrow |\det \Lambda|^2 = 1 \Rightarrow \det \Lambda = \pm 1$.

The physical case $\quad \det \Lambda = +1$.

$\det \Lambda = 1 \rightarrow$ Proper Lorentz transformations.

$\det \Lambda = -1 \rightarrow$ improper Lorentz $\quad$ " $\quad$.

Look at 00 component of $\quad \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} \eta_{\mu\nu} = \eta_{\alpha\beta}$.

$\eta_{\mu\nu} \Lambda^{\mu}{}_{0} \Lambda^{\nu}{}_{0} = \eta_{00} = -1$.

$-(\Lambda^{0}{}_{0})^2 + \sum_{i=1}^{3} (\Lambda^{i}{}_{0})^2 = -1$

$(\Lambda^{0}{}_{0})^2 = 1 + \sum_{i} (\Lambda^{i}{}_{0})^2 \geq +1$.

$(\Lambda^{0}{}_{0})^2 \geq +1 \quad \begin{cases} \rightarrow \Lambda^{0}{}_{0} \geq +1 & \text{orthochronos} \\ \rightarrow \Lambda^{0}{}_{0} \leq -1 & \text{non-orthochronos} \end{cases}$

Examples of Lorentz transformations

1) $\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \rightarrow$ Proper - non - orthochronos

$\rightarrow$ reflection.

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2) $t \to -t \qquad \vec{x} \to \vec{x}$.

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \to \text{improper} \to \text{non-orthochronos}$$

3) Physical Lorentz transformations →

→ Continousely connected to the identity →

→ ⓐ $\det \Lambda = 1 \leftarrow$ Proper.

ⓑ $\Lambda^{0}{}_{0} \geq 1 \leftarrow$ orthochronos

Examples: $\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad \Lambda^{\mu}{}_{\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$

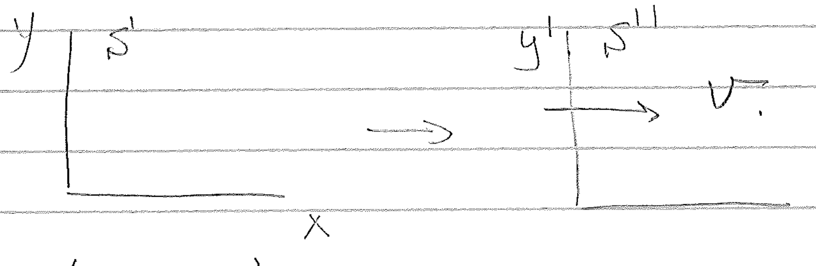
① Rotations $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & \vec{R} \end{pmatrix}$

$\det \Lambda = \det R$

$\det R = \pm 1$

$\det R = 1 \to$ Proper.

② boosts



$$ct' = \gamma(ct - \beta x) \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \beta = \frac{v}{c}$$

$$x' = \gamma(x - \beta ct)$$

$$y' = y$$

$$z' = z$$

OR

$$x'^0 = \gamma(x^0 - \beta x^1)$$
$$x'^1 = \gamma(-\beta x^0 + x^1)$$
$$x'^2 = x^2$$
$$x'^3 = x^3$$

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or $$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$\det \Lambda = \gamma^2 - \gamma^2\beta^2 = \frac{1}{(1-\beta^2)}(1-\beta^2) = 1.$$

$$\Lambda^0{}_0 = \gamma = \frac{1}{\sqrt{1-\beta^2}} \geq 1 \quad \text{as } \beta = \frac{v}{c} < 1.$$

The Poincare group.

consider the infinitesimal relativistic line element

(*) $$-ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2.$$

So far we discussed Rotation & boosts

which are the Lorentz transformations.

The most general transformations also include

translations. $X^{\mu} \to X'^{\mu} = X^{\mu} + a^{\mu}$

where a^{μ} is a constant 4-vector

obviously $dt' = dt \quad dx'^i = dx^i$.

we have:
- 4 - translations dt, dx, dy, dz,
- 3 - boost. $dt\,dx, dt\,dy, dt\,dz$
- 3 - Rotations $dx\,dy, dx\,dz, dy\,dz$.

These are the poincare transformations.

→ They form the Poincare group.

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Reprensentation of the Poincare group are labeled by:

1) spin → label of the Lorentz group.

2) mass

spin	Examples	spin	component	
		0	1	singlet
		½	2	Weyl spinor
		½	4	Dirac spinor = W-spinor + W spinor
		+1	4	vector

Relativistic energy and momentum.

Non relativistic energy-momentum relation: $E = \frac{P^2}{2m} = \frac{mv^2}{2}$

Relativistic " " " $\frac{E^2}{c^2} = \vec{P}\cdot\vec{P} + m^2c^2$

Relativistic $E = \gamma mc^2$ $\vec{P} = \gamma m\vec{v}$

Energy-momentum 4-vector $P^\mu = (P^0, P^1, P^2, P^3) = (\frac{E}{c}, P^1, P^2, P^3)$

$= (\frac{E}{c}, \vec{P}) = m\gamma(c, \vec{v})$

$\Rightarrow P_\mu P^\mu = \eta_{\mu\nu} P^\mu P^\nu = -\frac{E^2}{c^2} + \vec{P}\cdot\vec{P} = -m^2c^2 = P^2$

$P_\mu P^\mu = -m^2c^2$ → Lorentz scalar (invariant)

time intervals are not Lorentz invariant.

Proper time - time measured in the moving-frame relative to itself.

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consider a particle moving along the x-axis.

consider: $-ds^2 = -c^2 dt^2 + dx^2 = -c^2 dt^2 (1-\beta^2)$. $dx = \beta c dt$

Evaluate the interval in a frame moving with the particle.

$dx' = 0 \quad dt' = dt_p \rightarrow$ proper time.

$ds^2 = c^2 dt_p^2 \Rightarrow ds = c\, dt_p \Rightarrow \frac{ds}{c} = \underset{\text{proper time}}{dt_p}$

also $ds = c\,dt(\sqrt{1-\beta^2}) \Rightarrow \frac{dt}{ds} = \frac{\gamma}{c}$.

$ds \rightarrow$ Lorentz scalar $\rightarrow$ construct new Lorentz vectors.

velocity four vector: $u^\mu = c\frac{dx^\mu}{ds} = c\left(\underbrace{\frac{d(ct)}{ds}}_{\gamma}, \frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right)$

simplify? e.g. $\frac{dx}{ds} = \left(\frac{dx}{dt}\right)\left(\frac{dt}{ds}\right) = v_x \frac{\gamma}{c}$

$\Rightarrow \quad u^\mu = \gamma(c, v_x, v_y, v_z)$.

$\Rightarrow \quad p^\mu = m u^\mu$

any Lorentz 4-vector transforms under Lorentz transformations as x^μ,

$$p'^\mu = \Lambda^\mu{}_\nu\, p^\nu$$

For x-boost

$$\begin{pmatrix} E'/c \\ p_x' \\ p_y' \\ p_z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \gamma\left(\frac{E}{c} - \beta p_x\right) \\ \gamma\left(-\frac{\beta E}{c} + p_x\right) \\ p_y \\ p_z \end{pmatrix}$$

LST.12 | 5/7/09.2 | Abingdon | sunday |

Lorentz invariance in extra dimensions

$$-ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 + dw^2 \ldots$$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & & \\ & +1 & & & \\ & & +1 & & \\ & & & +1 & \\ & & & & +1 \end{pmatrix}$$

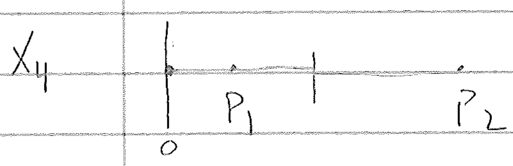
etc...

more components but otherwise the same.

compact extra dimensions.

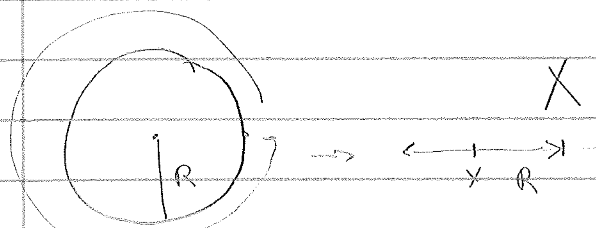
only 4 are observed.

$\Rightarrow$ Extra dimensions are compact - curled.



$X(P_2) = X(P_1) + 2\pi R n$

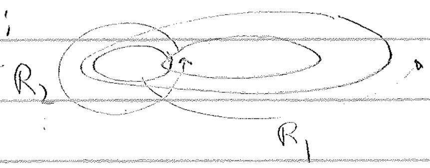
n - integer R - constant.

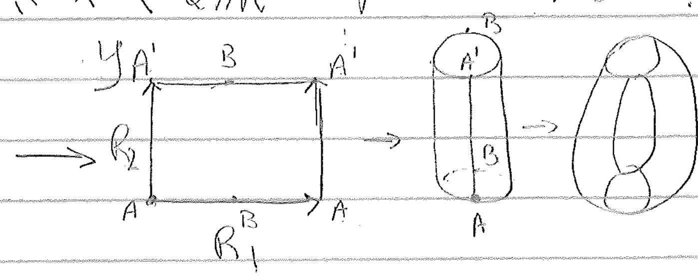


$X \sim X + 2\pi R n$, ← circle.

$0 \le X \le 2\pi R$ - fundamental domain.

Torus:





$X \sim X + 2\pi R_1 n$ $y \sim y + 2\pi R_2 n$

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consider the real line $-\infty \le x \le \infty$

impose the condition $x \sim -x$.

points are identified under the Z_2 reflection.

The fundamental domain is now $x \geq 0$.

$x = 0$ is a fixed point.

This is the orbifold R_1 / Z_2

The orbifold is singular at the fixed point.

Quantum mechanics and the square well

Planck constant expresses the relation between the energy of a photon and its angular frequency.

$$E = \hbar \omega.$$

it is the fundamental constant of quantum mechanics

it appears in the basic commutation relations of quantum mechanics.

$$[\hat{x}, \hat{p}] = i\hbar.$$

and the limit $\lim_{\hbar \to 0}$ gives the classical limit.

From $E = \hbar\omega$ its units are

$$[E] = [M] \cdot \frac{[L]^2}{[T]^2} = [\hbar] \cdot \frac{1}{[T]} \Rightarrow [\hbar] = [M] \frac{[L^2]}{[T]} = [E][T] = [S]$$

$m v^2$

[action] in Lagrangian mechanics

LST. 14 | 5/8/09.2 | Abingdon | Wednesday |

In Extra dimensions.

$$[X^i, P_j] = i\hbar\, \delta^i{}_j$$

where

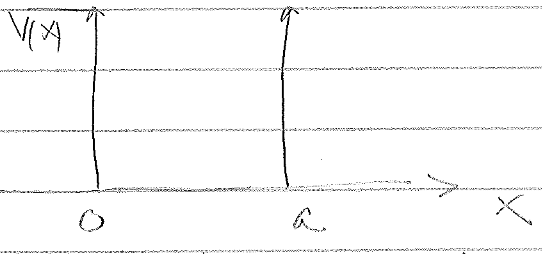
$$\delta^i{}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

consider the time-independent schrödinger eq. in 1-dimension.

$$-\frac{\hbar^2}{2m}\vec{\nabla}^2 \psi(x) + V(x)\,\psi(x) = E\,\psi(x)$$

In the case of of a square-well potential of infinite height.

$$V(x) = \begin{cases} 0 & x \in (0, a) \\ \infty & x \notin (0, a) \end{cases}$$



The boundary conditions imply that

$$\psi(x) = 0 \quad \text{for} \quad x \notin (0, a)$$

continuity of the wave function on the boundary implies that $\psi(0) = \psi(a) = 0$.

$\Rightarrow$ quantum mechanics of a particle confined to the segment $0 < x < a$.

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For $0 \leq x \leq a$. $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi = E\psi$ (*)

$\Rightarrow \psi(x) = A\sin\lambda x + B\cos\lambda x.$

$\psi(0) = 0 \Rightarrow B = 0.$

$\psi(a) = 0 \Rightarrow A\sin\lambda a = 0 \Rightarrow \lambda = \frac{k\pi}{a}$

$k = 1, 2, 3 \dots \infty$

sub into (*) $\Rightarrow +\frac{\hbar^2}{2m}\frac{k^2\pi^2}{a^2} A\sin\frac{k\pi}{a}x = EA\sin\frac{k\pi}{a}x$

$\Rightarrow E = \frac{\hbar^2 k^2\pi^2}{2ma^2}$

To find A impose $\int_{-\infty}^{\infty}|\psi|^2 dx = 1 = \int_{-\infty}^{\infty}\rho\, dx$

$\cos^2 - \sin^2 = \cos 2x$
$\cos^2 + \sin^2 = 1$

$= \int_0^a A^2\sin^2\frac{k\pi}{a}x\, dx = A^2\int_0^a \frac{1}{2}\left(1 - \cos\frac{2k\pi x}{a}\right) dx =$

$= \frac{A^2}{2}\left(x\Big|_0^a - \frac{a}{2k\pi}\sin\frac{2k\pi}{a}x\Big|_0^a\right) = \frac{A^2 a}{2} = 1$

$\Rightarrow A = \sqrt{\frac{2}{a}} \Rightarrow \psi(x) = \sqrt{\frac{2}{a}}\sin\frac{k\pi x}{a}.$

square well with EXTRA dimension.

Consider the case with an extra compact dimension.

$(x, y) \sim (x, y + 2\pi R).$

with

$$V(x, y) = \begin{cases} \infty & x \notin (0, a) \\ 0 & x \in (0, a) \end{cases}$$

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We will consider the case with $R << a$.

The Schrödinger eq. in two dimensions is:

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)+V(x,y)\psi(x,y)=E\psi(x,y).$$

using separation of variables.

$$\psi(x,y) = \sigma(x)\,\phi(y)$$

$$\Rightarrow -\frac{\hbar^2}{2m}\left(\sigma_{xx}\phi + \sigma\,\phi_{yy}\right) = E\,\sigma(x)\phi(y) \qquad x\in(0,a).$$

$$\Rightarrow -\frac{\hbar^2}{2m}\left(\frac{\sigma_{xx}}{\sigma}+\frac{\phi_{yy}}{\phi}\right) = E = \text{const}.$$

$$\Rightarrow -\frac{\hbar^2}{2m}\frac{\sigma_{xx}}{\sigma} = c_1 \qquad -\frac{\hbar^2}{2m}\frac{\phi_{yy}}{\phi} = c_2 \quad \text{with } c_1+c_2=E$$

We solve the two equations subject to the boundary conditions with $\phi(y) = \phi(y+2\pi R)$.

in x-direction we have as before

$$\sigma_k(x) = c_k \sin\frac{k\pi x}{a}$$

in y-direction |

$$\phi_\ell(y) = a_\ell \sin\frac{\ell y}{R} + b_\ell \cos\frac{\ell y}{R}.$$

for $\ell = 0$ we have.

$$\phi_0(y) = b_0$$

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Putting back the solutions into the equations for x and y

$$\Rightarrow E_{k,l} = C_1 + C_2 = \frac{\hbar^2}{2m}\left[\frac{k^2\pi^2}{a^2} + \frac{l^2}{R^2}\right]$$

new : For $l \neq 0$ the spectrum is doubly degenerate $\pm l$.

For $l = 0$ the energy states are the same as the 1-dimensional case.

Lowest new state: The lowest new energy level is for $l = 1$ $k = 1$.

$$E_{1,1} = \frac{\hbar^2}{2m}\left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{1}{R}\right)^2\right]$$

in the case with $R << a$

$$E_{1,1} \sim \frac{\hbar^2}{2m}\frac{1}{R^2}$$

This energy is comparable to the the k-level eigenstates of the one dimensional problem when

$$\frac{k\pi}{a} \sim \frac{1}{R} \longrightarrow k \sim \frac{1}{\pi}\frac{a}{R}.$$

Since $R << a$ k is very large.

$\Rightarrow$ The first new energy levels are at an energy far above those of the low lying original states.

$\rightarrow$ The extra dimension remains hidden provided that $R << a$.

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As we will see later strings introduce a new feature.

$$E \sim \frac{m^2}{R^2} + \frac{n^2 R^2}{\alpha'^2} \qquad (0)$$

strings can wrap around the compact dimension.

and create new states ~ wrapping modes.

For point particle → no wrapping modes $E \sim \frac{m^2}{R^2}$.

Elecromagnetism and gravitation in various dimensions

classical eletrodynamics | Maxwell's equation describe the dynamics of electric and magnetic fields in the presence of electric charges and currents.

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot B = 0 \rightarrow \text{no magnetic charges}$$

$$\vec{\nabla} \cdot E = \rho \rightarrow \text{electric field has charged sources}$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \rightarrow \text{moving charges create a magnetic field.}$$

$\vec{E}, \vec{B}$ are the three dimensional electric and magnetic fields

ρ, j are the electric charge and current densities.

Lorentz law: $\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$ → effect of E & B fields on a moving charged particle.

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since $\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla} \times \vec{A}$

because (using our summation convention and $\epsilon_{ijk} = \begin{cases} +1 & \epsilon_{123} \\ -1 & \epsilon_{213} \\ 0 & \epsilon_{113} \end{cases}$ etc., $i,j,k = 1,2,3$)

$$\left(\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A})\right) = \partial_i \epsilon_{ijk} \partial_j A_k = \epsilon_{ijk} \partial_i \partial_j A_k =$$

$$= -\epsilon_{jik} \partial_i \partial_j A_k = -\epsilon_{ijk} \partial_j \partial_i A_k =$$

$$= -\epsilon_{ijk} \partial_i \partial_j A_k \Rightarrow \epsilon_{ijk} \partial_i \partial_j A_k = 0.$$

in electrostatics. $\vec{\nabla} \times \vec{E} = 0 \Rightarrow \vec{E} = -\vec{\nabla}\phi$

in electrodynamics $\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial}{\partial t} \vec{\nabla} \times \vec{A}$ (From Maxwell eq.)

$$\Rightarrow \vec{\nabla} \times \left(E + \frac{1}{c}\frac{\partial A}{\partial t}\right) = 0$$

$$\Rightarrow \vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}.$$

$\Rightarrow$ we can describe E & M dynamics in terms of the potential functions $\vec{A}$ & ϕ.

Manifestly relativistic electrodynamics

$A^\mu = (\Phi, A^1, A^2, A^3) \rightarrow$ 4-vector.

$A_\mu = (-\Phi, +\vec{A})$.

From A_μ we create the electomagnetic field strength

$$F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)$$

$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right)$ $\quad \partial^\mu = \left(-\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right)$

we have,

$$F_{\nu\mu} = \partial_\nu A_\mu - \partial_\mu A_\nu = -(\partial_\mu A_\nu - \partial_\nu A_\mu) = -F_{\mu\nu}.$$

$\Rightarrow$ $F_{\mu\nu}$ is antisymmetric.

$\Rightarrow$ diagonal components of $F_{\mu\nu}$ vanish.

let i denote a spatial index i.e. $i = 1, 2, 3.$

Then $F_{0i} = \partial_0(+A_i) - \partial_i A_0 = \frac{1}{c}\frac{\partial}{\partial t}(+A^i) + \frac{\partial}{\partial x^i}\Phi = -E_i$

$$F_{ij} = \partial_i(+A_j) - \partial_j(+A_i) = +(\partial_i A_j - \partial_j A_i) = +(\vec{\nabla}\times\vec{A})_k = +B_k.$$

Hence,

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ +E_1 & 0 & +B_3 & -B_2 \\ +E_2 & -B_3 & 0 & +B_1 \\ +E_3 & +B_2 & -B_1 & 0 \end{pmatrix}.$$

$\rightarrow$ The electric and magnetic fields are encoded in the field strength tensor $F_{\mu\nu}$

gauge transformations: The potential A^μ can be changed, by gauge transformations

$$A_\mu \longrightarrow A'_\mu = A_\mu + \partial_\mu \epsilon(x^\mu)$$

where $\epsilon(x^\mu)$ is a scalar function of x^μ.

without affecting $F_{\mu\nu}$.

$$F_{\mu\nu} \longrightarrow F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = \partial_\mu(A_\nu + \partial_\nu\epsilon) - \partial_\nu(A_\mu + \partial_\mu\epsilon) = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}$$

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In terms of $\phi, \vec{A}$ we have.

$$\phi \rightarrow \phi' = \phi - \frac{1}{c}\frac{\partial \epsilon}{\partial t}.$$
$$\vec{A} \rightarrow \vec{A}' = \vec{A} + \vec{\nabla}\epsilon.$$

and $\vec{E}$ and $\vec{B}$ are unchanged.

i.e. $\vec{E}' = -\vec{\nabla}\phi' - \frac{1}{c}\frac{\partial \vec{A}'}{\partial t} = -\nabla\left(\phi - \frac{1}{c}\frac{\partial\epsilon}{\partial t}\right) - \frac{1}{c}\frac{\partial}{\partial t}\left(\vec{A} + \vec{\nabla}\epsilon\right)$

$= -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial}{\partial t}\vec{A} + \cancel{\frac{1}{c}\vec{\nabla}\frac{\partial}{\partial t}\epsilon} - \cancel{\frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\epsilon} = \vec{E}.$

and $\vec{\nabla}\times\vec{A}' = \vec{\nabla}\times(\vec{A} + \vec{\nabla}\epsilon) = \vec{\nabla}\times\vec{A} + \vec{\nabla}\times\vec{\nabla}\epsilon = \vec{\nabla}\times\vec{A} = \vec{B}.$

(because $\vec{\nabla}\times\vec{\nabla}\epsilon = \epsilon_{ijk}\partial_j\partial_k\epsilon = -\epsilon_{ikj}\partial_j\partial_k\epsilon = -\epsilon_{ikj}\partial_k\partial_j\epsilon = -\vec{\nabla}\times\vec{\nabla}\epsilon$)

Next we write Maxwell's equations in a manifestly Relativistic form.

consider | $T_{\lambda\mu\nu} = \partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda}$

$T_{\lambda\mu\nu}$ is identically zero

because | $T_{\lambda\mu\nu} = \partial_\lambda(\partial_\mu A_\nu - \partial_\nu A_\mu) + \partial_\nu(\partial_\lambda A_\mu - \partial_\mu A_\lambda) + \partial_\mu(\partial_\nu A_\lambda - \partial_\lambda A_\nu) = 0$

hence | $\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0.$

using $F_{\mu\nu} = -F_{\nu\mu}$ it follows that,

$T_{\lambda\mu\nu} = -T_{\lambda\nu\mu}$ $\quad T_{\lambda\mu\nu} = -T_{\mu\lambda\nu}$

i.e. $T_{\lambda\mu\nu}$ is antisymmetric under exchange of any two indecies.

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The source force Maxwell equations are obtained from $T_{\lambda\mu\nu} = \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} \equiv 0$

Since $T_{\lambda\mu\nu}$ is totally antisymmetric and $T_{\lambda\mu\nu} \equiv 0$.

$\Rightarrow$ 4 independent equations

e.g. $\lambda = 0 \quad \mu = i \quad \nu = j$.

$$T_{0ij} = \partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} =$$

$$= +\partial_t (+B_k) + \partial_i (+E_j) - \partial_j E_i =$$

$$= +\partial_t B_k + \epsilon_{kij} \partial_i E_j = 0$$

$$\Rightarrow (\nabla \times \vec{E})_k + \frac{\partial (\vec{B})_k}{\partial t} = 0.$$

FOR $\lambda = i \quad \mu = j \quad \nu = k$

$$T_{ijk} = \partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij}$$

take $i=1$ $j=2$ $k=3$ $\Rightarrow \partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12}$

$$= \partial_1 (+B_1) + \partial_2 (+B_2) + \partial_3 (+B_3) = 0$$

$$\Rightarrow (\vec{\nabla} \cdot \vec{B}) = 0$$

The two remaining Maxwell equations contain electric charges and currents.

introduce: $j^\mu = (c\rho, j^1, j^2, j^3) \rightarrow$ FOUR vector current density.

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$\rho \to$ charge density.

$\vec{J} = (j_1, j_2, j_3)$ current density.

introduce | $F^{\mu\nu} = \eta^{\mu\alpha} \eta^{\nu\beta} F_{\alpha\beta}$

Show (1) $F^{\mu\nu} = -F^{\nu\mu}$ $\quad F^{0i} = -F_{0i}$. $\quad F^{ij} = F_{ij}$.

2) $F^{\mu\nu} = (\partial^\mu A^\nu - \partial^\nu A^\mu)$

3) $F^{\mu\nu} = \begin{pmatrix} 0 & +E_1 & +E_2 & +E_3 \\ -E_1 & 0 & +B_3 & -B_2 \\ -E_2 & -B_3 & 0 & +B_1 \\ -E_3 & +B_2 & -B_1 & 0 \end{pmatrix}$.

The remaining Maxwell equations then take the form.

$\partial_\nu F^{\mu\nu} = \frac{\partial}{\partial x^\nu} F^{\mu\nu} = \frac{1}{c} j^\mu$ (H.W. 2)

In the absence of sources.

$\Rightarrow \partial_\nu F^{\mu\nu} = 0 \to \partial_\nu \partial^\mu A^\nu - \partial_\nu \partial^\nu A^\mu = 0.$

OR, $\partial^2 A^\mu - \partial^\mu (\partial \cdot A) = 0$.

where $\partial_\nu \partial^\nu = \partial^2$.

we can now use these equations to generalize Maxwell equations to any dimensions.

4D $\quad 0, 1, 2, 3. \quad \eta^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}$.

nD $\quad 0, 1, 2, \ldots, n \quad \eta^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & \ddots \end{pmatrix}_{n \times n}$

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Example: 3D space-time $\to$ 1+2 $\to$ $\begin{pmatrix} -1 & & \\ & +1 & \\ & & +1 \end{pmatrix}$

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y \\ +E_x & 0 & +B_z \\ +E_y & -B_z & 0 \end{pmatrix}.$$

in D dimension: $F^{0i} = -F_{0i} = +E_i \to$ electric field

$F_{ij} = -F_{ji} \Rightarrow$ magnetic field

in D-dimension E_i is a spatial vector $i = 1, .. D$

B_i is not a spatial vector.

Spheres in higher dimensions.

a 2-sphere in 3 Dimensions defined by the equation.

$$S^2(R) = X_1^2 + X_2^2 + X_3^2 = R^2$$

The volume enclosed by the sphere is a ball

$$B^3(R) = X_1^2 + X_2^2 + X_3^2 \leq R^2$$

where R is the Radius of the ball.

similarly in D-dimensions.

$$S^{D-1}(R) = X_1^2 + X_2^2 + \cdots + X_D^2 = R^2$$

$$B^D(R) = X_1^2 + X_2^2 + \cdots + X_D^2 \leq R^2.$$

$X_1^2 + X_2^2 = R^2$

the volume of a 1-sphere: $vol_1(S^1(R)) = 2\pi R$ (length)

" " " a 2-sphere: $vol(S^2(R)) = 4\pi R^2$ (area)

⋮

" " " a D-1 sphere $vol(S^{D-1}(R)) = vol(S^{D-1}) R^{D-1}$