OSU Notes
Construction Of The Fermionic Field

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Contents

Chapter 1

Plan Of The Lectures

The aim of these lectures is to provide a plan for constructing fermionic field models. The focus of these lectures is therefore on the construction of these models rather than the preliminary construction of superstring theories.

However, the construction of superstring theories has culminated in a set of rules derived by Antoniadis, Bachas and Kounnas as well as Kawai, Lewellen and Tye. This set of rules is our tool for constructing the realistic free fermionic field (FFF) models.
These notes will provide a 'flavor' of the derivation of these rules, but not the full derivations as they can be lengthy.

The plan is therefore

1. The ABK and KLT free fermionic field building
2. Construction of consistent vacua
3. Construction of realistic vacua
4. Derivation of the superpotential
5. Phenomenology and future directions

Chapter 2

The ABK and KLT Construction

String model building follows the following methodology:

1. Construct string vacua, consistent with the string consistency conditions
2. Extract the massless spectrum
   (For some purposes the massive spectrum is also of interest and can be extracted similarly)
3. Analyze the cubic level and higher order terms in the superpotential
4. Study flat F and D directions while imposing phenomenological criteria
5. Study the string characteristics that fix the phenomenological properties of the models
   $\Rightarrow$ Extract the general properties

1) Construction of consistent string vacua in FFF.

We will have a brief reminder of the basics of strings.
Strings are one dimensional objects (rather than 0 dimensional objects such as point particles) propagating in space and time.
If the world line is parameterized only by the proper time τ then the action is

$$S = -m \int ds = -m \int d\tau \sqrt{-g_{\mu\nu} \frac{dX^\mu}{d\tau} \frac{dX^\nu}{d\tau}} = \frac{1}{2} \int d\tau \left(\frac{1}{e} \dot{X}^2 - e m^2 \right) \quad (2.1)$$

where $\int ds$ is the integral over the world line and $\dot{X} = \frac{\partial X}{\partial \tau}$. Here e is interpreted as an 'einbein' for the one dimensional geometry of the world line which ensures that the action is invariant under reparameterizations of the proper time τ.

If the world line is parameterized by the proper time τ and an arbitrary parameter σ which is orientated along the world line, (i.e $X = X(\tau, \sigma)$) then the corresponding action is

$$
\begin{aligned}
S &= -T\int d\sigma d\tau\sqrt{-\det\ \frac{\partial X^{\mu}}{d\sigma^{\alpha}}\frac{dX^{\nu}}{\partial\sigma^{\beta}}\eta_{\mu\nu}} \\
&= -T\int d^2\sigma\big[(\dot{X}\cdot X')^2-(\dot{X}^2\cdot X'^2)\big]
\end{aligned}
\tag{2.2}
$$

where T is the string tension, $X' = \frac{\partial X}{\partial \sigma}$ and $\alpha, \beta = 0, 1$.
This can also be expressed in a linear form as

$$
S = \frac{1}{4\pi\alpha'}\int_{\Lambda g} d^2\sigma\sqrt{-\det\ g_{\alpha\beta}}\, g^{\alpha\beta}\partial_{\alpha}X^{\mu}\partial_{\beta}X^{\nu}\eta_{\mu\nu} \tag{2.3}
$$

where $g_{\alpha\beta}$ is the world sheet metric, $\eta_{\mu\nu}$ is the flat Minkowski metric and α' is the slope parameter. From this we can also infer that the string tension $T = \frac{1}{4\pi\alpha'}$.
The worldsheet metric $g_{\alpha\beta}$ is an auxiliary field that can be integrated out to find the Nambu-Goto (NG) action S_{NG}, which is

$$
S_{NG} = -\frac{1}{2\pi\alpha'} \qquad \big(\text{area of the worldsheet}\big)
$$

Constraints on S

a) Reparameterization invariance *i.e*

$$
\tau \to \tilde{\tau}(\tau) \qquad \text{for the world line parameterized only by } \tau
$$

$$
\begin{cases} \sigma \to \tilde{\sigma}(\sigma,\tau) \\ \tau \to \tilde{\tau}(\sigma,\tau) \end{cases} \qquad \text{for the world line parameterized by both } \tau \text{ and } \sigma
$$

b) Weyl rescaling

$$
g_{\alpha\beta}(\sigma,\tau) \to e^{f(\sigma,\tau)}g_{\alpha\beta}(\sigma,\tau)
$$

Due to this, we can fix the independent components of $g_{\alpha\beta}$ which implies that the 2 dimensional metric becomes $g_{\alpha\beta} \to \eta_{\alpha\beta}$.[1]
This can be interpreted as using Weyl rescaling to set the world sheet metric equal to the flat 2d metric $\eta_{\mu\nu}$. We will use this shortly to create the *flat gauge* action (equation (2.4)) which simplifies our theory considerably.
Choosing a metric of this form is known as selecting a *conformal gauge* which is a gauge that preserves angles locally.

If we consider the complex coordinates (which we introduce as they will become useful for writing some later expressions. We will also call these the world sheet coordinates)

$$
\begin{aligned}
z &= \tau + i\sigma \\
\bar{z} &= \tau - i\sigma
\end{aligned}
$$

then we can show explicitly how Weyl rescaling works. We begin by defining

$$
\begin{aligned}
g_{zz} &= g_{\bar{z}\bar{z}} = 0 \\
g_{z\bar{z}} &= g_{\bar{z}z} = \frac{1}{2}e^{\phi(z,\bar{z})}
\end{aligned}
$$

[1]For definition of Weyl invariance, see David Tong 'String Theory' p.20

where g is metric. When written in matrix form we obtain

$$g = \begin{pmatrix} 0 & \frac{1}{2}e^{\phi} \\ \frac{1}{2}e^{\phi} & 0 \end{pmatrix} \qquad \Rightarrow \det g = -\frac{1}{4}e^{2\phi}$$

where ϕ is some function on the world sheet that depends on z and $\bar{z}$ (note that it is equivalent to the $f(\sigma, \tau)$ in the Weyl rescaling definition above). Using the result of the determinant we find

$$\sqrt{-\det g_{\alpha\beta}} = \sqrt{+\frac{1}{4}e^{2\phi}} = \frac{1}{2}e^{\phi}$$

$$g^{\alpha\beta} = 2e^{-\phi}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sqrt{-\det g_{\alpha\beta}}\ g^{\alpha\beta} = \frac{1}{2}e^{\phi} \cdot 2e^{-\phi}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the Louiville mode drops from the action ($i.e$ the term $\sqrt{-\det g_{\alpha\beta}}\ g^{\alpha\beta}$ is removed from the action (2.3)).

The classical action is now invariant under the conformal transformations as we have removed the Louiville mode. This means the following transformations hold

$$Z \rightarrow f(Z)$$
$$\bar{Z} \rightarrow f(\bar{Z})$$

The conformal invariance of the classical action is one of the important properties of the perturbative string as string theories are described in terms of 2d conformal field theories.
The flat gauge action is a two dimensional conformal field theory of D-free bosons (where D is the number of dimensions).

$$S = \frac{1}{4\pi\alpha'}\int_{\Lambda_g} dz\, d\bar{z}\ \ \partial_z X^{\mu}\, \partial_{\bar{z}} X_{\mu} = 0 \tag{2.4}$$

The equation of motion from this action is a 2 dimensional wave equation

$$\left(\partial_{\sigma}^2 - \partial_{\tau}^2\right) X^{\mu}(\sigma, \tau) = 0 \tag{2.5}$$

which can also be written in complex coordinates with the form

$$\partial_z \partial_{\bar{z}}\ X^{\mu}(z, \bar{z}) = 0 \tag{2.6}$$

which indicates that $(\partial_z X^{\mu})$ is an analytic function of z.
This simply reflects the fact that the solution of the 2d wave equation can split arbitrarily into left and right moving solutions.

$$X^{\mu}(\sigma, \tau) = X_R^{\mu}(\tau + \sigma) + X_L^{\mu}(\tau - \sigma) \tag{2.7a}$$

$$X^{\mu}(z, \bar{z}) = X_R^{\mu}(z) + X_L^{\mu}(\bar{z}) \tag{2.7b}$$

where equation (2.7b) is written in complex coordinates.
For a closed string we have the periodicity condition

$$X^{\mu}(\sigma, \tau) = X^{\mu}(\sigma + \pi, \tau) \tag{2.8}$$

and (as an aside) we have the open strings periodicity condition

$$X^\mu(\sigma = 0; \pi) = 0$$

These solutions can then be expanded in Fourier modes

$$X^\mu_{L,R} = \frac{1}{2}X^\mu + \frac{1}{2}l^2 p^\mu(\tau \mp \sigma) + \frac{i}{2}l \sum \alpha_n^{\mu(n)}\, e^{-2in(\tau \mp \sigma)} \tag{2.9a}$$

$$\begin{aligned} \partial_z X^\mu &\sim \sum_n z^{(-n-1)}\, \alpha_n^\mu \\ \partial_{\bar{z}} X^\mu &\sim \sum_n \bar{z}^{(-n-1)}\, \tilde{\alpha}_n^\mu \end{aligned} \tag{2.9b}$$

where equation (2.9a) is the mode expansion of (2.7a) and (2.9b) are the mode expansions of (2.7b).
Classically, $\alpha_n^\mu, \tilde{\alpha}_n^\mu$ have the physical interpretation of being the amplitudes of the n^{th} oscillation mode. However, in a quantum theory $\alpha_n^\mu, \tilde{\alpha}_n^\mu$ are realised as creation and annihilation operators. It is these operators that define a Hilbert space.

2.1 Quantization of the Bosonic String

To properly quantize the theory with local gauge symmetries we must either introduce Fadeev-Popov ghost fields or perform canonical quantization with first-class constraints.

2.1.1 Canonical Quantization

First we will briefly consider the canonical approach. We begin this by imposing equal time commutation relations

$$\big[X^\mu(\sigma', \tau), X^\nu(\sigma, \tau)\big] = \eta^{\mu\nu}\delta(\sigma - \sigma')T^{-1} \tag{2.10}$$

If we vary the action with respect to the worldsheet metric and demand that the result equal zero, we get the 2d energy-momentum tensor. As $g_{\mu\nu}$ is an auxiliary parameter we have the constraint

$$\frac{\delta S}{\delta g_{\alpha\beta}} \;\Rightarrow\; T_{\alpha\beta} = 0 \tag{2.11}$$

in the *light cone* (*i.e* worldsheet) coordinates. Recall that the light cone coordinates are[2]

$$\begin{aligned} x^+ &\equiv \frac{1}{\sqrt{2}}(x^0 + x^1) \\ x^- &\equiv \frac{1}{\sqrt{2}}(x^0 - x^1) \end{aligned} \tag{2.12}$$

[2]For an in depth review of light cone coordinates, see Zwiebach - Section 2.3

In our complex coordinates we have $T_{z\bar{z}} = 0$ which follows from the tracelessness of the energy momentum tensor (this condition is equivalent to stating that there is conformal symmetry[3]). We therefore have the two constraint operators

$$T(z) = T_{zz} = -\frac{1}{2}\partial_z X^\mu \partial_z X_\mu \tag{2.13a}$$

$$\bar{T}(\bar{z}) = T_{\bar{z}z} = -\frac{1}{2}\partial_{\bar{z}} X^\mu \partial_{\bar{z}} X_\mu \tag{2.13b}$$

The energy momentum tensor can be expanded in terms of Fourier modes

$$T(z) = \sum_n z^{(-n-2)}\, L_n \tag{2.14}$$

and classically the Fourier modes L_n satisfy the Virasoro algebra[4]

$$\big[L_n, L_m\big] = (n-m)L_{m+n} \tag{2.15}$$

Quantum mechanically, $L_n, \tilde{L}_n$ become the harmonic oscillator's creation and annihilation operators. We can write this explicitly to show the form of these Fourier modes

$$L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_{n-m}\cdot\alpha_n$$

where it is more obvious that these modes correspond to the creation and annihilation operators.

We then have to consider the case where there is an *anomaly* in the Virasoro algebra. The motivation for considering this is that the energy momentum tensor (which depends on this Virasoro algebra, see equation (2.14)) in the quantum theory is generally not traceless, which is a condition we need[5]. We need this to be true as this is the condition for a conformal theory. We consider this anomaly by introducing a second term into the Virasoro algebra, which is done by

$$\big[L_n, L_m\big] = (n-m)L_{(n+m)} + \frac{c}{12}(n(n^2-1))\,\delta_{(n+m)} \tag{2.16}$$

Here, c is the central charge of the Virasoro algebra. In this case it is equal to the number of dimensions d of the target space, or equivalently the number of free scalar fields on the world sheet[6].

This anomaly arises due to the fact that the Louiville mode does not decouple when we quantize the theory, meaning we can not simply recover the condition that $g_{\mu\nu} \to \eta_{\mu\nu}$. This signals the break down of conformal invariance.

For the bosonic string we find that only in the case of $D = 26$ does the anomaly vanishes (*i.e* $c = 0$).

The constraint equation on the energy momentum tensor is then expressed in terms of Fourier modes. We should also note that there is a normal ordering

[3]See 'Introduction to Conformal Field Theory' - Blumenhagen p 19
[4]See notes Virasoro Modes and Algebra
[5]GSW Vol 1 p75
[6]Kitisis - 'String Theory in a Nutshell' - p30

ambiguity in L_0.[7]
The constraint equation (equations (2.13a) and (2.13b)) becomes the condition that the L_n and $\bar{L}_n$ annihiliate the physical state. This means (for the left movers)

$$\begin{aligned} &L_n \left|\text{Phys}\right\rangle = 0 \qquad \text{for } n > 0 \\ &\left(L_0 - a\right)\left|\text{Phys}\right\rangle = 0 \qquad \left(L_0 - \bar{L}_0\right)\left|\text{Phys}\right\rangle = 0 \end{aligned} \tag{2.17}$$

where there are similar condition for the right movers. The constant a is introduced here to resolve normal ordering ambiguities in the normal-ordered operator L_0. As the Fourier modes correspond to quantum operators, their ordering is important. By introducing this constant a we aim to resolve this problem, however we will leave it undetermined at this point.

We should note here the physical interpretations of these Virasoro generators. L_{-1} and $\bar{L}_{-1}$ generate translations in the plane, where as L_0 and $\bar{L}_0$ generate scaling and rotations. These generators are therefore performing conformal transformations.[8]

The spectrum at this level contains states with negative normals due to the time-like commution relation and the residual gauge freedoms. However, we can fix the remaining freedoms by considering the light cone gauge

$$X_{\pm} = \frac{1}{\sqrt{2}}\left(X^0 \pm X^{(D-1)}\right) \tag{2.18}$$

The spectrum is free of negative norm states when the light cone gauge is used, but it is not manifestly Lorentz invariant. It can be shown that it is only Lorentz invariant for D=26 and a=1.

2.1.2 Quantization Using the Fadeev-Popov Formalism

Alternatively, we can use the Fadeev-Popov formalism the quantize the theory in the covariant path integral formalism. This is what we will consider now.

By doing this we get a contribution c_{anomaly} from the ghost fields which is independent of the space-time dimensions. The c here is the same that was introduced in the Virasoro anomaly (equation (2.16)).
We find

$$c_{\text{gh}} = -26 \tag{2.19}$$

for the bosonic string. This means the total anomaly is

$$c_{\text{B}} = D \cdot 1 + (-26) \tag{2.20}$$

where c_{B} is the contribution for the bosonic string.

The mass of the string can be obtained in terms of the Fourier coefficients of the constraint equation

$$L_0 = \frac{1}{2}\sum_{-\infty}^{\infty} \alpha_{-n}\alpha_n = \frac{1}{2}\alpha_0^2 + \frac{1}{2}\sum_{n\neq 0} \alpha_{-n}\alpha_n = 0 \tag{2.21}$$

[7]For definition, see 'Lectures on String Theory' - David Tong - p31
[8]see 'Lectures on String Theory' - David Tong - p94

or by using the mass squared definition $M^2 = -p^2$

$$M^2 = 2\sum_{n=1}^{\infty}\big(\alpha_{-n}\alpha_n + \tilde{\alpha}_{-n}\tilde{\alpha}_n\big) \tag{2.22}$$

where we have used

$$\alpha_0^2 = \tilde{\alpha}_0^2 = \frac{1}{4}p^\mu p_\mu = -\frac{1}{4}M^2 \tag{2.23}$$

This can be seen in the footnote[9].
For the closed string, $L_0 = \tilde{L}_0$ due to last constraint in equation (2.17). Then we get

$$M^2 = -8a + 8\sum_{n=1}^{\infty}\alpha_{-n}\alpha_n = -8a + 8\sum_{n=1}^{\infty}\tilde{\alpha}_{-n}\tilde{\alpha}_n \tag{2.24}$$

where the equality is shown as the Virasoro constraint affecting the mass of the left and right movers. The factor of 8 here is significant as it means that mass squared is four times as large for closed strings as for the open string case. We also note that the introduction of the normal ordering constant a has introduced a shift in the mass squared spectrum.

The Hilbert space is obtained by acting with the bosonic oscillations on the vacuum. However, in order to completely describe the string state we must include another degree of freedom which is the center of mass momentum p^μ. The vacuum state with a center of mass momentum is defined as

$$|0, p^\mu\rangle = ie^{-p_\mu X^\mu}|0\rangle \tag{2.25}$$

For $n = 0$ we see that $M^2 < 0$. We can see this if we substitute the relation $M^2 = -p^2$ into the right hand side of equation (2.25). This implies that there are tachyons in the theory because there are scalar particles which travel in a space-like manner.
The first energy level above this is then

$$\alpha_{-1}^\mu\tilde{\alpha}_{-1}^\nu|0, p^\mu\rangle \tag{2.26}$$

by continuing in this way we can build the entire spectrum of physical states.

2.1.3 Introducing Supersymmetry

So far we have been discussing the bosonic string. As we have seen, it contains

- Tachyons
- No space-time fermions

therefore this implies a need for supersymmetry. We can introduce superpartners for the bosons X^μ by demanding that for each boson $X^\mu \to X^\mu, \psi^\mu$. The $\psi^\mu(z, \bar{z})$ here are D-Majorana world sheet fermions, where the D refers to the dimensions of the fermions.

[9] Zwiebach - 'A First Course in String Theory' - p252

The flat gauge action defined in equation (2.4) is therfore modified to include these superpartners. It now takes the form

$$S = \int d^2\sigma \big[\partial_\alpha X^\mu \partial^\alpha X_\mu - i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu\big] \tag{2.27}$$

where ρ^α are the 2d gamma matrices. These matrices are included due to the use of the Majorana fermions, for clarification see footnote[10].
We should note here that it is not immediately obvious that this will lead to spacetime fermions or spacetime supersymmetry as the fermion fields introduced here are actually still bosons with respect to the spacetime Lorentz group. It is also noted that they currently have no mass.

The gauge fixed action exhibits global worldsheet supersymmetry. Local worldsheet supersymmetry can also be exhibited by using the ungauged fixed action.

We can write the Majorana fermion in terms of it's two Weyl compenents[10]

$$\psi^\mu(z, \bar{z}) = \begin{pmatrix} \psi^\mu(z) \\ \bar{\psi}^\mu(\bar{z}) \end{pmatrix} \tag{2.28}$$

Then by using this and the light cone worldsheet coordinates, the action becomes

$$S = \frac{i}{\pi} \int d^2z \big(\psi_z \, \partial_{\bar{z}} \psi_z + \bar{\psi}_{\bar{z}} \, \partial_z \bar{\psi}_{\bar{z}}\big) \tag{2.29}$$

which explicitly shows that the left and right moving degrees of freedom are decoupled.

Similarly to the bosonic case, the fermion modes give rise to negative norm states. In the covariant path integral formulism the total fermionic ghost contribution is

$$c_{\text{fg}} = +11 \tag{2.30}$$

while each Majorana fermion gives $c = \frac{1}{2}D$, where D is the dimension.
For the superstring we get

$$c_{\text{total}} = -26 + 11 + D + \frac{D}{2} = -15 + \frac{3D}{2} \tag{2.31}$$

therefore to get $c_{\text{total}} = 0$ we find $D = 10$.

Similarly to the bosonic case, to construct the theory in which the negative norm states are decoupled we work in the light cone gauge in which only transverse excitations act on the non-degenerate vacua.

For the bosonic coordinates of a closed string we have the boundary conditions

$$X(\sigma, \tau) = X(\sigma + \pi, \tau) \tag{2.32}$$

For the Majorana coordinates we can have either periodic or anti-periodic boundary conditions for the left or right movers seperately

$$\psi_\pm(\sigma, \tau) = \pm\psi_\pm(\sigma + \pi, \tau)\big|^{R}_{NS} \tag{2.33}$$

[10] A detailed analysis of Dirac, Majorana and Weyl fermions can be found in the paper "Dirac, Majorana and Weyl fermions" - Palash B. Pal at http://arxiv.org/pdf/1006.1718v2.pdf

where R refers to the Ramond boundary conditions and NS refers to the Neveu-Schwarz boundary conditions. This gives rise to the mode expansions

$$\psi_-^\mu(\pi) = \psi_-^\mu(0) \qquad R: \ \psi_-^\mu = \sum d_n^\mu \, e^{\left(-2in(\tau-\sigma)\right)} \qquad \leftarrow \text{periodic} \tag{2.34a}$$

$$\psi_-^\mu(\pi) = -\psi_-^\mu(0) \quad NS: \ \psi_-^\mu = \sum b_r^\mu \, e^{\left(-2ir(\tau-\sigma)\right)} \quad \leftarrow \text{anti-periodic} \tag{2.34b}$$

where $n \in \mathbb{Z}$ and $r \in \mathbb{Z} + \frac{1}{2}$. There are also similar mode expansions for the left movers ψ_+^μ.

The equal time anticommutation relation for the fermionic coordinates gives the anticommutation relation for the fermionic R and NS oscillation modes

$$\{b_r^\mu, b_s^\nu\} = \eta^{\mu\nu}\delta_{r+s} \ \{d_m^\mu, d_n^\nu\} = \eta^{\mu\nu}\delta_{m+n} \tag{2.35}$$

In the case of NS boundary conditions the non-degenerate Fock space[11] created is unique because the lowest lying state is

$$b^\mu_{-\frac{1}{2}} \left|0\right\rangle \tag{2.36}$$

which transforms as a vector of the Lorentz group

$$\{b^\mu_{-\frac{1}{2}}, b^\nu_{+\frac{1}{2}}\} = \eta^{\mu\nu} \tag{2.37}$$

However, in the Ramond sector the zero modes obey the Dirac algebra

$$\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu} \tag{2.38}$$

which commutes with the mass operator.
Therefore the zero modes are proportional to the Dirac gamma states and the states

$$d_0^\mu \left|0\right\rangle \tag{2.39}$$

transform in the spinorial representation of the Lorentz group, therefore generating spacetime fermions.

For a single real R-fermion, we can think of this as a doubly degenerate state

$$d_0 \left|0\right\rangle \ ; \ \left|0\right\rangle \tag{2.40}$$

There are some other points that we should note

1. Similar to the energy momentum tensor, we have the worldsheet supercurrent which is obtained by varying the action with respect to the 2d gravitino

$$\begin{aligned} T_F(z) &= \psi^\mu \, \partial_z X_\mu(z) \\ \bar{T}_F(\bar{z}) &= \bar{\psi}^\mu \, \partial_{\bar{z}} X_\mu(\bar{z}) \end{aligned} \tag{2.41}$$

2. The mass shell condition is modified

$$\big(L_0 - 1\big) \left|\text{Phys}\right\rangle = 0 \rightarrow \big(L_0 - \frac{1}{2}\big) \left|\text{Phys}\right\rangle = 0 \tag{2.42}$$

[11] The Fock space is an algebraic construction used to construct a space of quantum states of an unknown number of identical particles from a single particle Hilbert space

3. We define a fermion number operator

$$(-1)^{N_{NS}} \qquad (-1)^{N_R} \tag{2.43}$$

which roughly counts the number of fermions acting on the non-degenerate ground state $|0\rangle_{NS}$ and the number of R-zero modes acting on the non-degenerate Ramond vacua.

2.2 Construction of String Vacua

The next important symmetry we must discuss is *modular invariance*. However, allow us to first start describing the construction of string vacua in the fermionic formulation and describe the modular invariance throughout this procedure.

Vanishing of the conformal anomaly gives the constraint

$$\begin{aligned} c_{\text{total}} &= c_{\text{bg}} + c_{\text{fg}} + c_{X^\mu} \cdot D + c_{\psi_\mu} \cdot D \\ &= -26 + 11 + D + \frac{D}{2} \end{aligned} \tag{2.44}$$

from which we can set $c_{\text{total}} = 0$, therefore implying $D = 10$ (recall that the $\frac{D}{2}$ factor is present due to the fermions being Majorana). We therefore get the ten dimensional superstring theory.

Due to the decoupling of the left and right moving modes for the closed string, we can cancel the left and right moving anomalies seperately.

In this case we can impose worldsheet supersymmetry in the left moving sector while the right moving sector is left as bosonic. This means for the central charges c we have

$$\begin{aligned} c_L &= -26 + 11 + D + \frac{D}{2} = 0 \qquad \Rightarrow D = 10 \\ c_R &= -26 + D \qquad\qquad\quad \Rightarrow D = 26 \end{aligned} \tag{2.45}$$

When a string theory is constructed in this way, it is called the *heterotic string construction*.

The cancellation of these anomalies requires that we 'compactify'. This involves mounting 16 right moving components on a flat torus which has a fixed radia. This is necessary as we can not have different spacetime dimensions for the left and right movers as this is unphysical. We therefore have to remove the 16 'extra' components from the bosonic sector.

For each coordinate that is compactified in this way, there is a generated *vertex operator* for a $U(1)$ space time current.[12]

We find that there are only two allowed choices of group that permit this. These groups are $E_8 \times E_8$ and $SO(32)$. These turn out to be the only anomaly free gauge groups that couple to $\mathcal{N} = 1$ supergravity in 10 dimensions, but this will become clearer when discussing the fermion construction.

We then compactify six dimensions of space onto a Calabi-Yau manifold in order

[12] In the vertex the α is a 'charge' and in string theory it is interpreted as the spacetime momentum along the $X(z, \bar{z})$ spacetime direction. See 'Introduction to Conformal Field Theory - Blumenhagen and Plauschinn' p51-53

to preserve the $\mathcal{N} = 1$ supersymmetry in 4D, which gives the representation of the group E_6 in 4D.

Alternatively, we can use the *Narain construction.* This construction treats all the bosonic coordinates equally and compactifies them on a flat torus with a Lorenzian even self dual lattice[13].

This construction gives a wider range of gauge groups that the theory can use in four dimensions and $\mathcal{N} = 4$ supersymmetry. The orbifolding technique can then be used to recover $\mathcal{N} = 1$ supersymmetry from the $\mathcal{N} = 4$ case. We will not cover how this is done in these notes, as the method is lengthy. We will instead continue by covering the fermionic construction.

However one remark should be made about the Narain construction. As in the Calabi-Yau case, we have a set of moduli that control the shape and size of the 6D compactified space.

2.2.1 Fermionic Construction

We will now begin considering the fermionic construction. Rather than identifying the degrees of freedom needed to cancel the conformal anomaly as spacetime dimensions, we can interpret them as free fermions which propagate on the string world sheet. This allows us to formulate the theory directly in four dimensions.

We have

$$\begin{aligned} c_L &= -26 + 11 + D + \frac{D}{2} + N_{f_L} \cdot \frac{1}{2} = 0 \\ c_R &= -26 + D + N_{f_R} \cdot \frac{1}{2} = 0 \end{aligned} \tag{2.46}$$

If we then include that $D = 4$ for our four dimensional spacetime then we find

$$\begin{aligned} N_{f_L} &= 18 \\ N_{f_R} &= 44 \end{aligned} \tag{2.47}$$

Therefore, we need 18 left moving real Majorana fermions and 44 right moving Majorana-Weyl fermions to cancel the left and right moving conformal anomalies. The left moving fermions must be Majorana as they must have mass (which is intrinsic in the definition of a Majorana fermion). The right movers must be Majorana-Weyl as these can be either massive or massless, therefore allowing our formulation to include massless bosons such as the photon.

A remark is in order here. The fermionic and bosonic constructions are entirely equivalent. In the fermionic construction we are working with free fields on the world sheet. The equivalence of the two cases follows from the equivalence of fermions and bosons in two dimensions. This can be seen explicitly by defining

$$\begin{aligned} e^{+iX} &= y + i\omega \\ e^{-iX} &= y - i\omega \end{aligned} \tag{2.48}$$

which allow us to go back to the bosonic construction. In these definitions, the $e^{i\pm X}$ factors represent the bosons and the $y \pm iw$ represent the fermions. When these are considered while using the vertex operators we can see they

[13]See file titled 'Lattices'

are equivalent as they both give a related 'conformal weight' when Operator Product Expansions (OPEs) are considered.[14]

The free fermion field is formulated at a fixed point in moduli space. If we wish to move away from the fixed point we have to include the fermionic world-sheet interations that preserve the conformal invariance. This is not particuarly well developed but while the equivalence between the fermionic and bosonic descriptions may not be known in the general case, they are entirely equivalent.

In the Polyakov picture, string theory is formulated as a perturbative sum over the path integral of the string worldsheet. The string worldsheet then defines a genus-g-Riemann surface (e.g for the tree level, g=0 and the Riemann surface is a sphere, for one loop g=1 and the Riemann surface is a torus).
Using the conformal invariance we can describe the string states as vertex operators on the genus-g-Riemann surface. However, we have to make sure that we are only integrating over physically inequivalent paths. This is necessary as integrating over physically equivalent paths leads to overcounting of identical physical states in the partition function.

The partition function is defined as the one loop vacuum to vacuum amplitude. Therefore we are integrating on a one dimensional Riemann surface *i.e* a torus.

The free fermions are world sheet fermions that can propagate around the non-contracting loops of the torus which can therefore pick up a phase

$$f \to -e^{-i\pi\alpha(f)}f \tag{2.49}$$

where f is the fermion and $\alpha(f)$ is the phase.
This forms the boundary conditions of the world sheet fermions as the fermions have a periodicity associated with each non-contractible loop. We will cover this in more detail in the section on spin structures.

2.2.2 Lecture 2

Let us remember that our goal is to identify a class of string compactificiations that are as realistic as possible. In the case that string theory is relevent, our hope is that such models will be relevant. On the other hand we are developing the methodology needed to confront string theory with experimental data. If string theory if relevent in Nature then our models will guide in trying to learn about the Planck scale dynamics.

In the light cone gauge we have the following world sheet content. (Here we will introduce a notation that will become clearer later).

[14]See 'Introduction to Conformal Field Theory - Blumenhagen and Plauschinn' - Section 2.9-

Left moving:

$$\begin{array}{lll} X_L^\mu(z) & \mu=1,2 & \text{The two transverse coordinates} \\ \psi_L^\mu & \mu=1,2 & \text{Their fermionic partners} \\ \chi_L\, y_i\, w_i & \text{18 internal real fermions} & \end{array} \tag{2.50}$$

Right moving:

$$\begin{array}{lll} X_R^\mu(\bar{z}) & \mu=1,2 & \text{Two transverse coordinates} \\ \phi_a^{1,\ldots,44}(\bar{z}) & \text{44 internal real fermions} & \end{array} \tag{2.51}$$

We have to specify the bounday conditions of all the world sheet fermions.
We specify these boundary conditions in basis vectors[15] with 64 entries that will be used to expand the models and the partition function. We need 64 entries as we have two fermionic contributions from ψ_L^μ as $\mu = 1, 2$, 18 contributions from the internal real left moving fermions and 44 contributions from the internal real right moving fermions.
When we expand the partition function we have to include all possible boundary conditions for all physically inequivalent paths, or physically inequivalent tori. We then have to integrate over all inequivalent tori which specify distinct paths to obtain the partition function (which is free from overcounting).

In the Polyakov approach to string theory a string amplitude is calculated by the path integral

$$\begin{aligned} A_n &= \sum_{g=0}^{\infty} A_n^{(g)} \\ &= \sum_{g=0}^{\infty} \int \mathcal{D}h\mathcal{D}X^\mu \int d^2z_1 \ldots d^2z_n \; V_1(z_1,\bar{z}_1)\ldots V_n(z_n,\bar{z}_n) \end{aligned} \tag{2.52}$$

where A_n is the string amplitude, g is the genus of Riemann surface, h is the worldsheet metric and V_i are vertex operators of extremal string states.
Due to the symmetry of the action, we must again ensure that we only sum over physically inequivalent paths.

We find the string topologies by using conformal invariance to map from the string world sheet to the Riemann surface.[16] They are mapped by

$$\begin{gathered} \text{Tree level} \mapsto \text{sphere} \\ \text{One loop} \mapsto \text{torus} \end{gathered}$$

and so forth.

At the tree level all of the reparameterizations are local, where as at higher levels further constraints will arise.

[15] Where the basis vectors are linearly independent vectors that can be linearly combined to completely describe any other vector in a vector space.

[16] We perform this mapping because we find it much easier to mathematically describe a Riemann sphere/torus as opposed to trying to describe the actual form of the string world sheet. It is conformal invariance that makes this possible.

2.3 The Partition Function and Modular Invariance

2.3.1 The Partition Function

In order to find these new constraints, it is instructive to consider the one loop amplitudes with no external states. This is just the one loop partition function, which is isomorphic to the torus.
The analysis here is done in analogy to quantum statistical mechanics, where the partition function is calculated by considering the periodic temperature. Here we are considering the time coordinate as a complex temperature τ.

The partition function (Z) is found by summing over all physical states that can propagate around the loop, then integrating over the inequivalent tori.

$$Z(\beta,\theta) = \sum_{s\in\mathcal{H}} \langle s|\, e^{i\theta\mathcal{P}} e^{-2\pi\beta H}\, |s\rangle \tag{2.53}$$

where $\mathcal{H}$ is the Hilbert space, $\mathcal{P}$ is the momentum operator, H is the Hamiltonian and $|s\rangle$ is some state that can be created and annihilated from the vacuum.
The first exponential is the spacial propagation and the second exponential is the time propagation.

$$Z(\beta,\theta) = \operatorname*{Tr}_{\mathcal{H}} e^{i\theta\mathcal{P}} e^{-2\pi\beta H} \tag{2.54}$$

We recall that the translations are generated by the zero modes of the energy momentum tensor[17]

$$\begin{aligned} H &= L_0 + \bar{L}_0 - \frac{1}{24} \\ P &= L_0 + \bar{L}_0 \end{aligned} \tag{2.55}$$

where the $1/24$ is the normal ordering ambiguity. By combining $\tau = i\beta - \frac{\theta}{2\pi}$ this can be rewritten as

$$Z(\tau) = q^{-\frac{1}{48}}\ \bar{q}^{-\frac{1}{48}} \operatorname*{Tr}_{\mathcal{H}} q^{L_0}\ \bar{q}^{\bar{L}_0} \tag{2.56}$$

where $q = e^{i2\pi\tau}$.

We know how L_0 acts on the Fock space and we can calculate this. We still have to specify the boundary conditions for each worldsheet fermion. The total partition function will then be the product of

$$Z_F(\tau) = \prod_{i=1}^{64} Z_i \begin{bmatrix}\theta\\ \beta\end{bmatrix} (\tau) \tag{2.57}$$

where the individual $Z_i \begin{bmatrix}\theta\\ \beta\end{bmatrix}$ depend on the boundary conditions in the 'time' (β) and 'spacial' (θ) directions. The product of 64 contributions is due to the 64 basis vectors needed to consider all the fermions.

The space choice fixes $\psi^\mu(z)$ ($i.e$ the space fermions) to be R/NS. For the NS 'time' boundary conditions we have

$$Z^{NS}_{NS}(\tau) = \operatorname*{Tr}_{NS} q^{L_0 - \frac{1}{48}} \tag{2.58a}$$

[17]See David Tong 'String Theory Notes' Section 6.4.2

$$Z_{NS}^{R}(\tau) = \operatorname*{Tr}_{R} q^{L_0 - \frac{1}{48}} \tag{2.58b}$$

and for the R 'time' we have

$$Z_{R}^{NS}(\tau) = \operatorname*{Tr}_{NS} (-1)^F q^{L_0 - \frac{1}{48}} \tag{2.59a}$$

$$Z_{R}^{R}(\tau) = \operatorname*{Tr}_{R} (-1)^F q^{L_0 - \frac{1}{48}} \tag{2.59b}$$

2.3.2 Modular Invariance

Now we come to the modular invariance. To understand what modular invariance is, we first note that we can map the torus to the complex plane. This is done by cutting the torus along its two noncontractible loops.

INSERT DIAGRAM

We can then define a complex parameter

$$z = \sigma_1 + i\sigma_2 \tag{2.60}$$

where the two coordinates σ_1, σ_2 are periodic with a length λ_1 and λ_2 respectively. The torus can then be specified on the complex plane by

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with

$$z \Leftrightarrow z + n_1\lambda_1 + n_2\lambda_2 \qquad n_i \in z \tag{2.61}$$

which specified the torus. Alternatively we see that the torus is R_z moded by the lattice. We can see that if we take $\lambda_1' = a\lambda_1$ and $\lambda_2' = a\lambda_2$ then we get the same torus as this is simply a rescaling in the complex plane.

The parameter that specifies inequivalent tori is then

$$\tau = \frac{\lambda_2}{\lambda_1} \tag{2.62}$$

This complex parameter then embodies the two real degrees of freedom inherent in torodial geometry and can either be called the *complex structure* or the *modular parameter*. However there are still tori which are conformally equivalent.

Suppose we take

$$\begin{pmatrix} \lambda_2' \\ \lambda_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda_2 \\ \lambda_1 \end{pmatrix} \tag{2.63}$$

with arbitrary $a, b, c, d \in \mathbb{Z}$.
Then

$$\tau \rightarrow \tau' = \frac{a\tau + b}{c\tau + d} \tag{2.64}$$

we then ask what the conditions are on τ' such that it describes the same tori as τ.[18]

The new torus is defined by the identification

$$z \Leftrightarrow z + n_1'\lambda_1' + n_2'\lambda_2' \qquad (n_1', n_2') \in \mathbb{Z} \tag{2.65}$$

we get

$$z \Leftrightarrow z + (n_1'd + n_2'b)\lambda_1 + (n_1'c + n_2'a)\lambda_2 \tag{2.66}$$

[18] For a proof of this equation, see file 'Proof of Equation 2.64'

The two tori will be the same provided that[19]

$$\begin{pmatrix} n_2 \\ n_1 \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} n_1' \\ n_2' \end{pmatrix} \tag{2.67a}$$

$$\begin{pmatrix} n_2' \\ n_1' \end{pmatrix} = \frac{1}{(ad-bc)} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} n_2 \\ n_1 \end{pmatrix} \tag{2.67b}$$

is true, with $n_2, n_1 \in z$. The relation (2.67b) is simply the inverse of the relation (2.67a). The two tori are identical if

$$a, b, c, d \in \mathrm{SL}(2, \mathbb{Z}) \tag{2.68}$$

as this group has the necessary conditions on a, b, c, d so that equations (2.67a) and (2.67b) are true[20]. To get the correct contributions we have to integrate over all the conformally inequivalent tori.

The fundamental domain of the modular group[21] is given by

$$\mathcal{F} \equiv \big\{ \tau \qquad |\tau| \geq 1 \qquad -\frac{1}{2} \leq |\tau| \leq \frac{1}{2} \big\} \tag{2.69}$$

INSERT DIAGRAM

and to get the contribution of all inequivalent tori we have to integrate over this fundamental domain and require that the partition function is invariant under modular transformations (if the partition function was not invariant under these transformations, we would not be able to use the modular transformations to ensure that the tori lies within the fundamental domain with the same contributing partition function).

In addition, we have to require that the partition function does not depend on the parameterization of the tori. This is related to the fact that the string action is invariant under reparameterization of the string worldsheet. It is also related to the fact that in two dimensions the metric is conformally equivalent to the flat metric. However, in the one loop amplitude not all the transformations are continuously connected to the identity. This means we have to require that the partition function is invariant under the modular transformations which are 'spanned' by

$$\tau \to -\frac{1}{\tau} \tag{2.70a}$$

$$\tau \to \tau + 1 \tag{2.70b}$$

which are the important transformations. Requiring invariance under them will lead to a set of constraints on the allowed boundary conditions. It is also these transformations that allow us to ensure that the torus is within the fundamental domain.

These are the consistency constraints derived by the ABK and KLT constructions.

[19]See file 'Proof of 2.67

[20]See 'Introduction to Conformal Field Theory - Blumenhagen & Plauschinn' p116

[21]see David Tong String notes p144

2.3.3 Implementation of ABK Constraints and Spin Structures

We will now see how these constraints are implemented.
For every world sheet fermion we have to specify the transformation around the two non-contractible loops of the torus. For real fermions, these can only be periodic or anti-periodic *i.e* R/NS[22].

So we have four possibilities for the propagation around these loops decribed by $f \to -e^{i\pi\alpha(f)} f$.

$$Z_f = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \ , \ Z_f = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ , \ Z_f = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ , \ Z_f = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{2.71}$$

These are called the *spin structures* of the fermion on the torus. More generally, we can also combine two real fermions to form a complex fermion by

$$f = \frac{1}{\sqrt{2}} f_1 + i f_2 \quad , \quad \bar{f} = \frac{1}{\sqrt{2}} f_1 - i f_2 \tag{2.72}$$

In this case, the boundary conditions may be complex

$$f \to -e^{-i\pi\alpha(f)} f \qquad \text{with} \quad \alpha(f) \in (-1, 1] \tag{2.73}$$

However, we will only consider the derivation for real fermions here.

The partition function is then given by

$$Z = \int \frac{d\tau d\bar{\tau}}{[Im(\tau)]^2} \, Z_B^2(\tau, \bar{\tau}) \sum_{\text{spin structure}} C \begin{pmatrix} a \\ b \end{pmatrix} Z_{\text{long},3/2} \begin{bmatrix} a_\psi \\ b_\psi \end{bmatrix} \sum_{f=1}^{64} Z_f \begin{bmatrix} \alpha(f) \\ \beta(f) \end{bmatrix} \tag{2.74}$$

where Z_B is the transverse coordinate contribution to the partition function, the $C \begin{pmatrix} a \\ b \end{pmatrix}$ are phases (and $a, b = 0$ or 1) which are yet to be calculated, the $Z_{\text{long},3/2}$ refers to the partition function contribution from the gravitinos ($3/2$) in the longitudinal direction.
The rules are derived by requiring that the partition function is invariant under the modular transformations

$$\tau \to -\frac{1}{\tau} \qquad\qquad (S \text{ Channel})$$

$$\tau \to \tau + 1 \qquad\qquad (T \text{ Channel})$$

which is what we will now consider.

The measure $\frac{d\tau d\bar{\tau}}{\tau_I^2}$ is modularly invariant by itself. Also the term

$$Z_B = \frac{1}{|\tau_2|^2 |\eta(\tau)|^2} \tag{2.76}$$

is modular invariant and is the contribuition of the transverse coordinates. Here, $\eta(\tau)$ is the Dedekind eta function and is defined as

$$\eta(\tau) = q^{1/24} \prod_n (1 - q^n) \tag{2.77}$$

[22] Imaginary boundary conditions exist for complex fermions, *i.e* they pick up a phase of $\pm i$

where $q = e^{2\pi i\tau}$.

The partition function is simply the sum over all the string states, massless and massive, that we have to include when propagating the string around a closed loop. For example, the $Z\begin{bmatrix}0\\0\end{bmatrix}$ spin structure is given by

$$\begin{aligned} Z\begin{bmatrix}0\\0\end{bmatrix} &= C\begin{pmatrix}0\\0\end{pmatrix} \mathrm{Tr}\, e^{2\pi i\tau H_{NS}} \\ &= C\begin{pmatrix}0\\0\end{pmatrix} \mathrm{Tr}\, q^{H_{NS}} \\ &= C\begin{pmatrix}0\\0\end{pmatrix}\ q^{-1/24}\mathrm{Tr}\, q^{\sum_{r=\frac{1}{2}}^{\infty}\ rb_{-r}b_r} \\ &= C\begin{pmatrix}0\\0\end{pmatrix}\ q^{-1/24}\prod_r\sum_{N_r} q^{rN_r} \\ &= C\begin{pmatrix}0\\0\end{pmatrix}\ q^{-1/24}\prod_{n=1}^{\infty}\left(1+q^{n-\frac{1}{2}}\right) \end{aligned} \tag{2.78}$$

This is the grand partition function for an ideal Fermi gas with energy levels $E_r = r$.[23]
We now define a new modular function ϑ by[24][25]

$$\vartheta_3 = \eta(\tau)q^{-1/24}\prod_{n=0}^{\infty}\left(1+q^{n+\frac{1}{2}}\right)\left(1+q^{n+\frac{1}{2}}\right) \tag{2.79}$$

It can be shown by simple rearrangment that

$$\begin{aligned} Z\begin{bmatrix}0\\0\end{bmatrix} &= \frac{\vartheta_3^{1/2}(\tau)}{\eta^{1/2}(\tau)} \\ &= \mathrm{Tr}\left[e^{i\tau H_{NS}}\right] \end{aligned} \tag{2.80}$$

Similarly, for the other spin structures we find

$$Z_F\begin{bmatrix}0\\1\end{bmatrix} = \sqrt{\frac{\vartheta_4(\tau)}{\eta}} = \mathrm{Tr}\ \left[(-1)^F e^{i\tau H_{NS}}\right] \tag{2.81a}$$

$$Z_F\begin{bmatrix}1\\0\end{bmatrix} = \sqrt{\frac{\vartheta_2(\tau)}{\eta}} = \mathrm{Tr}\ \left[e^{i\tau H_{R}}\right] \tag{2.81b}$$

$$Z_F\begin{bmatrix}1\\1\end{bmatrix} = \sqrt{\frac{\vartheta_1(\tau)}{\eta}} = \mathrm{Tr}\ \left[(-1)^F e^{i\tau H_{R}}\right] \tag{2.81c}$$

The fermionic partition function is then just a product of the left and right moving ϑ functions (which depend on z and $\bar{z}$).

[23] For proof, see file 'Proof of 2.78'

[24] See String Theory and Particle Physics - An Introduction to String Phenomenology - Ibanez and Urunga - p577

[25] See 'Introduction to Conformal Field Theory - Blumenhagen - equation (4.36)

2.3.4 Modular Transformations of the Spin Structures

We can now determine the effect of the modular transformations given by equation (2.70a) and (2.70b) on the spin structures.

The effect of the S channel $\tau \to -\frac{1}{\tau} = -\frac{\lambda_1}{\lambda_2}$ is just to interchange $\lambda_1 \leftrightarrow \lambda_2$. We will now just state the result that these modular transformations have on the spin structures which is obtained using the Poisson resummation[26] and the theta functions.

For $\tau \to \tau + 1$

$$\eta \to e^{i\pi/12}\eta \tag{2.82a}$$

$$\vartheta_1 \to e^{i\pi/4}\vartheta_1 \tag{2.82b}$$

$$\vartheta_2 \to e^{i\pi/4}\vartheta_2 \tag{2.82c}$$

$$\vartheta_3 \leftrightarrow \vartheta_4 \tag{2.82d}$$

and for $\tau \to -\frac{1}{\tau}$

$$\eta \to (-i\tau)^{\frac{1}{2}}\eta \tag{2.83a}$$

$$\frac{\vartheta_1}{\eta} \to e^{-i\pi/2}\frac{\vartheta_1}{\eta} \tag{2.83b}$$

$$\frac{\vartheta_2}{\eta} \leftrightarrow \frac{\vartheta_4}{\eta} \tag{2.83c}$$

$$\frac{\vartheta_3}{\eta} \to \frac{\vartheta_3}{\eta} \tag{2.83d}$$

The crucial point is that now the fermionic partition function is a product of the spin structures of 64 fermions. Performing the modular transformations $\tau \to -\frac{1}{\tau}$ and $\tau \to \tau + 1$ will take us from one spin structure to another, *i.e* it will take us from one product of ϑ_i functions to another.

Modular invariance requires that both of the spin structures related under these transformations must contribute to the partition function with an equal weight, which is

$$C\begin{pmatrix}\vec{\alpha}\\ \vec{\beta}\end{pmatrix}$$

This condition will lead us to some constraints, which we will consider below. The one loop fermionic partition function is then

$$\sum_{\text{Spin Structures}} C\begin{pmatrix}\vec{\alpha}\\ \vec{\beta}\end{pmatrix} Z\begin{pmatrix}\vec{\alpha}\\ \vec{\beta}\end{pmatrix} \tag{2.84}$$

for all the spin structures allowed by modular invariance.

For example, we consider the modular transformation $\tau \to \tau + 1$

$$\left(\vartheta_3^{1/2}\right)^n\Big|_{\alpha=0,\beta=0} \xrightarrow{\tau\to\tau+1} \left(\vartheta_4^{1/2}\right)^n\Big|_{\alpha=0,\beta=1} \quad , \quad i.e\; Z\begin{bmatrix}0\\0\end{bmatrix} \to Z\begin{bmatrix}0\\1\end{bmatrix} \tag{2.85a}$$

$$\left(\vartheta_4^{1/2}\right)^m\Big|_{\alpha=0,\beta=1} \xrightarrow{\tau\to\tau+1} \left(\vartheta_3^{1/2}\right)^m\Big|_{\alpha=0,\beta=0} \quad , \quad i.e\; Z\begin{bmatrix}0\\1\end{bmatrix} \to Z\begin{bmatrix}0\\0\end{bmatrix} \tag{2.85b}$$

[26]Basic Concepts of String Theory - Blumenhagen p257

$$(\vartheta_2^{1/2})^l\big|_{\alpha=1,\beta=0} \xrightarrow{\tau\to\tau+1} (\vartheta_2^{1/2})^l\big|_{\alpha=1,\beta=0}\cdot(e^{i\pi/4})^{l\cdot 1/2} \qquad , i.e\ Z\begin{bmatrix}1\\0\end{bmatrix}\to e^{i i\pi/8}\ Z\begin{bmatrix}1\\0\end{bmatrix}$$
$$= (\vartheta_2^{1/2})^l\cdot e^{i\frac{\pi}{8}\sum_l \alpha_f} \tag{2.85c}$$

$$\vartheta_1 \equiv 0 \tag{2.85d}$$

where $\vartheta_1 = 0$ because the spin structure that it is associated with is already modular invariant under both the transformations.
We also have similar expressions for the right movers by taking the complex conjugate of the initial ϑ functions.
In the expression for ϑ_2 the sum appears in the exponential due to counting l number of fermions which have the boundary conditions α_f.

To ensure modular invariance we must impose the following condition on the coefficients C[27]

$$C\begin{pmatrix}\alpha\\\beta\end{pmatrix} = -\exp(i\frac{\pi}{8}\sum\alpha_f)C\begin{pmatrix}\alpha\\\alpha+\beta+1\end{pmatrix} \tag{2.86}$$

where the negative sign is present due to the η's and the sum is explicitly $\sum = \sum_{\text{left-movers}} - \sum_{\text{right-movers}}$. We also have

$$\sum\ \alpha_f = 0 \bmod 8 \tag{2.87}$$

Our ϑ_1 expression is as follows

$$\vartheta_1(Z,\tau) = 2e^{i\pi\tau/4}\sin\pi Z\prod_{n=1}^{\infty} f(\tau,z,n) \tag{2.88}$$

Now we will state how the ϑ functions transform under the modular transformation $\tau\to-\frac{1}{\tau}$

$$\begin{pmatrix}0\\0\end{pmatrix}^n : \left(\frac{\vartheta_3^{1/2}(\tau)}{\eta^{1/2}}\right)^n \longrightarrow \begin{pmatrix}0\\0\end{pmatrix}^n : \left(\frac{\vartheta_3^{1/2}}{\eta^{1/2}}\right)^n \tag{2.89a}$$

$$\begin{pmatrix}0\\1\end{pmatrix}^m : \left(\frac{\vartheta_4^{1/2}}{\eta^{1/2}}\right)^m \longrightarrow \begin{pmatrix}1\\0\end{pmatrix}^m : \left(\frac{\vartheta_2^{1/2}}{\eta^{1/2}}\right)^m \tag{2.89b}$$

$$\begin{pmatrix}1\\0\end{pmatrix}^l : \left(\frac{\vartheta_2^{1/2}}{\eta^{1/2}}\right)^m \longrightarrow \begin{pmatrix}0\\1\end{pmatrix}^m : \left(\frac{\vartheta_4^{1/2}}{\eta^{1/2}}\right)^l \tag{2.89c}$$

$$\begin{pmatrix}1\\1\end{pmatrix}^p : \left(\frac{\vartheta_1^{1/2}}{\eta^{1/2}}\right)^p \longrightarrow \begin{pmatrix}1\\1\end{pmatrix}^p : e^{-i\pi/4}\left(\frac{\vartheta_1^{1/2}}{\eta^{1/2}}\right)^p \tag{2.89d}$$

We must also impose another condition on C in order to maintain modular invariance, this is[28]

$$C\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \exp(i\frac{\pi}{4}\sum\alpha_f\cdot\beta_f)C\begin{pmatrix}\beta\\\alpha\end{pmatrix} \tag{2.90}$$

[27] See "Four dimensional Superstrings - ABK" - equation 2.6a
[28] See "Four dimensional Superstrings - ABK" - equation 2.6b

where

$$\vec{\alpha} \cdot \vec{\beta} = 0 \bmod 4 \tag{2.91}$$

where $\vec{\alpha} \cdot \vec{\beta}$ is a Lorenzian product, *i.e*

$$\sum_{\text{left}} \alpha_f \cdot \beta_f - \sum_{\text{right}} \alpha_f \cdot \beta_f \tag{2.92}$$

This is how the ABK rules are derived.
In additon, the modular invariance imposes the GSO projection on the physical spectrum.

The entire partition function can then be generated by specifying a set of boundary condition basis vectors

$$B = \{\vec{b}_1, \ldots, \vec{b}_n\} \tag{2.93}$$

and all the terms in the partition function are of the form

$$Z \begin{pmatrix} \vec{\alpha} \\ \vec{\beta} \end{pmatrix} \tag{2.94}$$

where

$$\vec{\alpha} = \alpha_1 \vec{b}_1 + \ldots + \alpha_n \vec{b}_n \tag{2.95a}$$

$$\vec{\beta} = \beta_1 \vec{b}_1 + \ldots + \beta_n \vec{b}_n \tag{2.95b}$$

For models with only NS/R boundary conditions the possible coefficients are

$$\alpha_i = 0, 1 \qquad \text{and} \qquad \beta_i = 0, 1 \tag{2.96}$$

where the basis vectors b_i and the coefficients $C \begin{pmatrix} b_i \\ b_j \end{pmatrix}$ are subject to the consistency constraints set out by the ABK rules.[29][30]

2.3.5 Ensuring a Well Defined Supercurrent

There is one more important constraint that should be mentioned. This important constraint has lead to some confusion in the construction of $SO(10)$ GUT models. The constraint is that worldsheet supersymmetry must be preserved. This means that the supercurrent T_F must be uniquely defined (up to a sign) under the transformation of the world sheet fermions under the specified boundary conditions in all sectors. Again, it is sufficient to ensure this for the basis vectors.

Recall that for the heterotic string the supercurrent has the form (from equation (2.41))

$$T_F = \psi^\mu \partial X_\mu \qquad\qquad \text{where } \mu = 1, \ldots, 8 \tag{2.97}$$

[29] See "4d Fermionic Superstrings with Arbitrary Twists - Antoniadis and Bachas" - Equations B1-B5

[30] Also see Chapter 3

In the fermionic construction, it will take the form

$$T_F = \psi^\mu \partial X_\mu + f_{abc}\psi^a\psi^b\psi^c \tag{2.98}$$

where f_{abc} are the structure functions of some semi-simple lie group of dimension 18.

In the models that are being described, the group is $SU(2)^6$. The 18 internal left moving fermions are then grouped into 6 representations in the adjoint of $SU(2)$. The supercurrent takes the form

$$T_F = \psi^\mu \partial X_\mu + \sum_{i=1}^{6} \chi_i y_i w_i \tag{2.99}$$

Requiring that the supercurrent is well defined then imposes that every product $\chi_i y_i w_i$ transforms with the same sign as $\psi^\mu \partial X_\mu$. This means that if $\psi^\mu \to -\psi^\mu$ we have

$$\chi_i y_i w_i \to -\chi_i y_i w_i$$

Therefore, the following boundary conditions are valid for each χ, y, w are

$$(\chi, y, w): \qquad (1,1,0) \qquad (0,1,1) \qquad (1,0,1) \qquad (0,0,0) \tag{2.100}$$

This is because for each boundary that is 0 (NS) there is a negative contribution, so the three boundary conditions must contain an odd number of 0's in order to contribute an overall negative sign.
By the same reasoning, if $\psi^\mu \to +\psi^\mu$ then the valid boundary conditions are

$$(\chi, y, w): \qquad (1,0,0) \qquad (0,1,0) \qquad (0,0,1) \qquad (1,1,1) \tag{2.101}$$

and $\chi, y, w \to \chi, y, w$. However, in this case we need an even number of NS (0) boundary conditions.
This ensures that we have a well defined world sheet supercurrent.

It would seem that we are done with the prerequisit. However, we aren't quite yet, but we are almost there. The fun only begins in constructing real models. Fortunately for us ABK summerized their rules very nicely.

We could have just stated the ABK rules amd forgot all about the preliminaries but it is useful to see where they come from.

The next thing we should know is that every complex fermion generates a world sheet current. This world sheet current produces the Cartan generators of the 4-dimensional gauge group. The charges with respect to these are given by

$$Q(f) = \frac{1}{2}\alpha(f) + F(f) \tag{2.102}$$

where $\alpha(f)$ is the boundary condition of a complex fermion and $F(f)$ is the fermion number.

The fermion number is given by (± 1) for a fermion acting on the non-degenerate vacuum and (-1) for its complex conjugate.

For the Ramond vacua there are two degenerate vacua denoted by $|\pm\rangle$ with fermion numbers

$$\begin{aligned} F: \qquad & |+\rangle = (0)\,|0\rangle \\ F: \qquad & |-\rangle = -1\,|+\rangle \end{aligned} \tag{2.103}$$

There is one more type of world sheet operator in the models that will be described. These are obtained by combining a real left moving with a real right moving fermion. They generate the conformal field theory of Ising operators with the field operator of the Ising model. This will become clearer when we discuss specific examples.

Chapter 3

ABK Rules and GSO Projections

We will now summarize the ABK rules and how GSO projection works so that we can start constructing real models.

A model is defined by specifying two ingredients

1. A set of boundary condition basis vectors
2. The one loop phases $C\begin{pmatrix} b_i \\ b_j \end{pmatrix}$ for all intersection of the basis vectors

3.1 Summary of the ABK Rules

We will now write down the explicit rules of the ABK models. This is our starting point for the construction of actual models.

1. Basis vectors $\{b_1, \ldots, b_n\}$

Each of these basis vectors consists of a set of boundary conditions for each fermion which can be written as

$$b_i = \{\alpha(\psi^{\mu}_{1,2}), \ldots, \alpha(w^6)|\alpha(\bar{y}^1), \ldots \alpha(\bar{\Phi}^8)\} \tag{3.1}$$

where $\alpha(f)$ is the phase defined by $f \to -e^{i\pi\alpha(f)}f$.
The basis vectors form an additive group Ξ which is defined by

$$\Xi = \sum_{i=1}^{n} m_i b_i \qquad\qquad \text{where } m_i = 0, \ldots, N_i - 1$$

where $N_i b_i = 0 \mod 2$.
Any spin structures that contribute to the partition function are pairs of elements in Ξ.

3.1.1 Rules on Basis Vectors

We should first recap that 1 implies a periodic (Ramond) boundary and 0 implies an anti-periodic (NS) boundary. Also in ABK we have $f \to -e^{i\pi\alpha(f)}f$.

1.
$$\sum m_i b_i = 0 \qquad \text{if and only if } \forall m_i = 0 \mod N_i \tag{3.2}$$
where N_i is the smallest positive integer where $N_i b_i = 0$[1].

2.
$$N_{ij} b_i b_j = 0 \mod 4 \tag{3.3}$$
where N_{ij} is the least common multiplier of b_i and b_j.

3.
$$N_i b_i b_i = 0 \mod 8 \tag{3.4}$$

4. Number of real fermions must be even

5.
$$b_1 = \mathbb{1} \qquad \text{(and more generally } \mathbb{1} \in \Xi) \tag{3.5}$$
where $\mathbb{1}$ is the basis vector which includes boundary conditions for all fermions.

In the above, we defined the scalar product between the basis vectors as

$$b_i \cdot b_j = \left\{ \sum_{\text{Complex Left}} + \frac{1}{2} \sum_{\text{Real Left}} - \left(\sum_{\text{Complex Right}} + \frac{1}{2} \sum_{\text{Real Right}} \right) \right\} b_i(f) b_j(f) \tag{3.6}$$

where $b_i(f)$ and $b_j(f)$ are the fermion numbers that count each real fermion once and each complex fermion minus once.

3.1.2 Rules on One-loop Phases

1.
$$C\binom{b_i}{b_j} = \delta_{b_i} e^{\frac{2\pi i}{N_j} n} = \delta_{b_j} e^{\frac{2\pi i}{N_i} m} e^{\frac{i\pi}{2} b_i \cdot b_j} \tag{3.7}$$

2.
$$C\binom{b_i}{b_i} = -e^{\frac{i\pi}{4} b_i \cdot b_i}\, C\binom{b_i}{\mathbb{1}} \tag{3.8}$$

3.
$$C\binom{b_i}{b_j} = e^{\frac{i\pi}{2} b_i \cdot b_j} \binom{b_j}{b_i}^* \tag{3.9}$$

4.
$$C\binom{b_i}{b_j + b_k} = \delta_{b_i} C\binom{b_i}{b_j} C\binom{b_i}{b_k} \tag{3.10}$$

where
$$\delta_{b_i} = e^{i\pi b_i(\psi^\mu)} = \begin{cases} -1 & b_i(\psi^\mu) = 1 \\ +1 & b_i(\psi^\mu) = 0 \end{cases} \tag{3.11}$$

This index ensures the correct spacetime statistics for spacetime fermions and bosons.

[1] We should note that for our cases N_i will always be 2. This is because we are working in $\Xi = \mathbb{Z}_2 \oplus \ldots \oplus \mathbb{Z}_2$ *i.e* the $\mathbb{Z}_2$ orbifold

3.1.3 Matching (Virasoro) Condition

The basis spans a finite additive group $\Xi = \sum m_i b_i$ for $m_i = 0, \ldots, N_i - 1$. For every sector α for which $\alpha \in \Xi$ there is a corresponding Hilbert space. This Hilbert space is obtained by acting with the fermionic oscillators (denoted by N_L and N_R) on the vacuum.
The states in this space have to satisfy the Virasoro conditions[2], this means they must have the same mass in the left and right sectors

$$M_L^2 = -\frac{1}{2} + \frac{\alpha_L \cdot \alpha_L}{8} + N_L = -1 + \frac{\alpha_R \cdot \alpha_R}{8} + N_R = M_R^2 \tag{3.12}$$

where α_L, α_R correspond to the left and right parts of α respectively and N_L,N_R are the total left and right oscillator numbers which act on the vacuum $|0\rangle_\alpha$.
For periodic fermions there is a double degenerate spinorial vacua ($|\pm\rangle$)

$$\nu_\alpha 0_\alpha \qquad ; \qquad |\pm\rangle$$

where ν_α is the frequency of the oscillators and is given for a fermion f and its conjugate f^* by

$$\nu_f = \frac{1+\alpha(f)}{2} \qquad\qquad \nu_{f^*} = \frac{1-\alpha(f)}{2} \tag{3.13}$$

The frequencies and the fermionic oscillators are related by

$$N_L = \sum \nu_L = \sum_{\substack{f \\ L-osc}} \nu_f + \sum_{\substack{f^* \\ L-osc}} \nu_{f^*}$$
$$N_R = \sum \nu_R = \sum_{\substack{f \\ R-osc}} \nu_f + \sum_{\substack{f^* \\ R-osc}} \nu_{f^*}$$

For the NS vacuum

$$\alpha(f) = 0 \qquad \Rightarrow \qquad \nu_f = \nu_{f^*} = \frac{1}{2}$$

3.1.4 Fermion Number

The fermion number is denoted by $F(f)$ and has the following properties

$$F(f) = \begin{cases} +1 & \text{for } f \\ -1 & \text{for } f^* \end{cases} \tag{3.14}$$

This will enter into some of our descriptions in the following sections, such as the $U(1)$ charges and the GSO projection.

3.1.5 $U(1)$ Charges

When the $U(1)$ gauge group is realised by a free fermion we have the corresponding charge

$$Q(f) = \frac{1}{2}\alpha(f) + F(f) \tag{3.15}$$

[2]For this section, see 'Classification of Free Fermionic Models - Marc-Olivier Renou' - p18

3.2 GSO Projection

The GSO projection is defined by the equation

$$e^{i\pi b_j \cdot F_\alpha} |s\rangle_\alpha = \delta_\alpha C \begin{pmatrix} \alpha \\ b_j \end{pmatrix}^* |s\rangle_\alpha \tag{3.16}$$

where

$$\alpha \in \Xi \qquad\qquad b_j \in \text{Basis}$$

and $|s\rangle_\alpha$ is a state in the sector $\alpha \in \Xi$. In the argument of the exponential we have used the definition

$$b_j \cdot F_\alpha = \left(\sum_{\text{Left}} - \sum_{\text{Right}} \right) b_j(f) F_\alpha(f) \tag{3.17}$$

The states $|s\rangle$ that do not satisfy this equation are *projected out* and no longer contribute to the partition function.
We should make a remark about the GSO projections. The GSO projection also arises by modular invariance.
The partition function simply counts the spectrum at all masses. When we expand the partition function for a sector α we can take it as a sum over its intersection with other sectors.

$$Z = \sum_\alpha \sum_\beta C \begin{pmatrix} \alpha \\ \beta \end{pmatrix} Z \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{3.18}$$

This means that in the sum there may be cancellations between different parts. These cancellations will be reflected in the spectrum. For example, $\mathcal{N} = 4$ SUSY has

$$\underbrace{\left[\vartheta_3^4 - \vartheta_2^4 - \vartheta_4^4 \right]} \vartheta_3^6 \bar{\vartheta}_3^{22}$$

where the underbraced term equals zero for $\mathcal{N} = 4$ SUSY.
When we construct the Hilbert space (of the models) by using the basis vectors, the GSO projections incorporate those cancellations. Therefore, they are just a result of the modular invariance of the partition function.

Chapter 4

Constructing Models

4.1 Single Basis Vector Model $B = \mathbb{1}$

Let us now finally construct models. We will begin with the simplest one with a single basis vector.

$$B = \{\vec{1}\} = \{\mathbb{1}\} \qquad \rightarrow \text{required by consistency}$$

In this case we only have two sectors

$$\{\vec{1}, 2 \cdot \vec{1} = \vec{0}\}$$

where $\vec{0}$ implies the NS boundary conditions.

The rules are trivially satisfied, to see how see the footnote[1].

In general the even number of real fermions means that if we pair two real fermions they will have the same boundary conditions in all sectors (basis vectors).

For phenomenology, we are only interested in massless states as we want to create states with low energy (*i.e* not at the unobtainable Planck energy scale). We must also ensure that there are no states with $m < 0$ which lead to a tachyon.
In this model we have two sectors
<u>$\vec{1}$ sector</u>

$$M_L^2 = -\frac{1}{2} + \frac{10}{8} + N_L = -\frac{1}{2} + \frac{5}{4} + N_L = \frac{3}{4} + N_L > 0 \tag{4.1}$$

The factor of 10 comes from the definition

$$\alpha_L \cdot \alpha_L = \frac{1}{2} \sum_{\text{Real Left}} = \frac{20}{2} = 10$$

where we used the definition from equation (3.6) and that there are 20 left moving fermions from equation (2.50).
For the right movers we have a similar expression

$$M_R^2 = -1 + \frac{44}{8} + N_R = -1 + \frac{11}{2} + N_R = \frac{9}{2} + N_R > 0 \tag{4.2}$$

[1] See "Building b=I First Model"

Equation (4.1) means this sector contains no massless states as $M_L^2 > 0$ is always true. Also equation (4.2) shows $M_R^2 > 0$ and therefore there are no massless states for the right movers either.

<u>NS sector</u>

$$M_L^2 = -\frac{1}{2} + \frac{0}{8} + N_L = -1 + \frac{0}{8} + N_R = M_R^2 \tag{4.3}$$

where the zero term comes from the fact that all the fermions are included in the basis vector $\mathbb{1}$, meaning there are no fermions in the NS sector.

The possible oscillators of the world sheet fermions are

$$\nu_{f,f^*} = \frac{1 \pm 0}{2} = \frac{1}{2} \tag{4.4}$$

So we need a combination of:

- One left moving fermionic oscillator

and either

- Two right moving fermionic oscillators
- One right moving bosonic oscillator

to get a massless state. Or we need one fermionic right moving oscillator to get a tachyonic state.[2]

4.1.1 Notation

We should remark on the notation that is introduced to describe these states. We will use the following notation, which is often found in the papers

<u>Left movers</u>

$$\psi^\mu_{1,2}(\chi_1 y_1 w_1)(\chi_2 y_2 w_2)(\chi_3 y_3 w_3)(\chi_4 y_4 w_4)(\chi_5 y_5 w_5)(\chi_6 y_6 w_6)$$

In this expression, the $\psi^\mu_{1,2}$ and all χ_i are complex, where as y_i and w_i are all real.

<u>Right movers</u>

$$\{\bar{y}_1\bar{w}_1\ \bar{y}_2\bar{w}_2\ \bar{y}_3\bar{w}_3\ \bar{y}_4\bar{w}_4\ \bar{y}_5\bar{w}_5\ \bar{y}_6\bar{w}_6\}\{\bar{\psi}_{1,\ldots,5}\ \bar{\eta}_1\ \bar{\eta}_2\ \bar{\eta}_3\ \bar{\phi}_{1,\ldots,8}\}$$

All the terms in the first brackets are real and all the terms in the second bracket are complex. The use of this notation will become more obvious while proceeding.

4.1.2 Massless States in the NS Sector

We therefore have the following massless states in the NS sector

- $$\psi^\mu_{1/2}\,\partial\bar{X}^\nu_1\ |0\rangle_{NS} \tag{4.5}$$

  where $\psi^\mu_{1/2}$ is the left moving fermion and $\partial\bar{X}^\nu_1$ is the bosonic creation operator (right moving boson). The bosonic states correspond to the graviton, dilaton and the antisymmetric tensor.

[2] See file 'Massless and Tachyon States in NS Sector'

- $$\psi^{\mu}_{1/2}\ \bar{\phi}^{a}_{1/2}\ \bar{\phi}^{b}_{1/2}\ \ |0\rangle_{NS} \qquad \{a,b\}=1,\dots,44 \tag{4.6}$$

  This state contains one left moving fermionic oscillator and two right moving fermionic oscillators. The right movers correspond to the gauge bosons in the adjoint representation of $SO(44)$.

- $$\{\chi^{i}_{1/2}, y^{i}_{1/2}, w^{i}_{1/2}\}\ \partial\bar{X}^{\mu}_{1}\ \ |0\rangle_{NS} \qquad \{i=1,\dots,6\} \tag{4.7}$$

  Here $\{\chi^{i}_{1/2}, y^{i}_{1/2}, w^{i}_{1/2}\}$ is the left moving fermionic oscillator and $\bar{\partial}X^{\mu}_{1}$ is the right moving bosonic oscillator. This state corresponds to the gauge bosons in the adjoint representation of $SU(2)^6$.

- $$\{\chi^{i}_{1/2}, y^{i}_{1/2}, w^{i}_{1/2}\}\ \bar{\phi}^{a}_{1/2}\ \bar{\phi}^{b}_{1/2}\ \ |0\rangle_{NS} \tag{4.8}$$

  This state contains one left moving fermionic oscillator and two right moving fermionic oscillators. This state corresponds to the scalars in the adjoint representation of the gauge group $SU(2)^6 \times SO(44)$.

We also have the tachyonic states which contains one right moving fermion

$$\bar{\phi}^{a}_{1/2}\,|0\rangle_{NS} \tag{4.9}$$

which has $M^2 = -\frac{1}{2}$.

4.1.3 GSO Projection of $B = \mathbb{1}$ Model

We now have to perform the GSO projections for each state in this model to see which states survive.

First we will define some results for the spin structures which will be useful for GSO projecting our states found in the previous section. Using rule (1) defined in equation (3.7) we find

$$C\begin{pmatrix} NS \\ NS \end{pmatrix} = \delta_{NS} e^{i\frac{2\pi 0}{0}} = \delta_{NS} = 1 \tag{4.10}$$

where $\delta_{NS} = 1$ due to equation (3.11) as $b_i(\psi^{\mu}) = NS = 0$.
Also by using rule (1), we find

$$C\begin{pmatrix} NS \\ b_j \end{pmatrix} = \delta_{b_j} \underbrace{e^{i\frac{2\pi m_{NS}}{N_{NS}}}}\underbrace{e^{i\frac{\pi b_j\cdot NS}{2}}} = \delta_{b_j} = \delta_{NS} e^{i\frac{2\pi n_j}{N_{b_j}}} \tag{4.11}$$

where the underbraced terms equal one. The first underbrace equals one as $m_{NS} = 0$ by definition as this is the NS sector and the second underbrace equals one because in the exponential $b_j \cdot NS = b_j \cdot 0 = 0$.
The useful identity we have obtained here is

$$\boxed{C\begin{pmatrix} NS \\ b_j \end{pmatrix} = \delta_{b_j}} \tag{4.12}$$

We can now continue to perform the GSO projection for our states. We will first consider the states in the massless spectrum by considering the GSO projection for a general state $|s\rangle$. For our additive group Ξ in this model, we have $\alpha = NS$ and $b_j = 1$ and therefore

$$\begin{aligned} e^{i\pi b_j \cdot F_\alpha}\,|s\rangle_\alpha \quad \longrightarrow \quad e^{i\pi \mathbb{1}\cdot F_{NS}}\,|s\rangle_{NS} &= \delta_{NS}\; C\binom{NS}{\mathbb{1}}^*\,|s\rangle_{NS} \\ &= \delta_{NS}\;\delta_{\mathbb{1}}\,|s\rangle_{NS} \\ &= \delta_{\mathbb{1}}\,|s\rangle_{NS} \\ &= -\,|s\rangle_{NS} \end{aligned} \tag{4.13}$$

To obtain the second line we used the boxed equation (4.12), to obtain line three we used the fact that $\delta_{NS} = +1$ and to obtain the final result we used $\delta_{\mathbb{1}} = -1$. These results for the δ's are true due to equation (3.11).

We can now perform this calculation for a specific state instead of our generic state $|s\rangle$. We will consider the first massless state of one left moving fermion and one right moving boson which describes the graviton, dilaton and the anti-symmetric tensor and see if this survives the GSO projection. We find

$$\begin{aligned} e^{i\pi \mathbb{1}\cdot F_{NS}}\left(\psi^{\mu}_{1/2}\,\partial\bar{X}^{\nu}_{1/2}\,|0\rangle_{NS}\right) &= \delta_{\mathbb{1}}\,\left(\psi^{\mu}_{1/2}\,\partial\bar{X}^{\nu}_{1/2}\,|0\rangle_{NS}\right) \\ &= -\,\psi^{\mu}_{1/2}\,\partial\bar{X}^{\nu}_{1/2}\,|0\rangle_{NS} \end{aligned} \tag{4.14}$$

(In general we find that the state $\psi^{\mu}_{1/2}\,\partial\bar{X}^{\nu}_{1/2}\,|0\rangle_{NS}$ always survives because the identity $C\binom{NS}{b_j} = \delta_{b_j}$ is true).
We also find

$$e^{i\pi b_j(\psi^\mu)F_{NS}(\psi^\mu)} = e^{i\pi b_j(\psi^\mu)} = \delta_{b_j}$$

which is true because the state that we are considering here $(\psi^\mu\partial\bar{X}^\nu\,|0\rangle)$ only contains the fermion ψ^μ, therefore giving the left hand side of this equation. The second exponential is true as $F_{NS}(\psi^\mu) = +1$ due to equation (3.14) and finally due to the spacetime spin statistics index (equation (3.11)) we find the answer. This means we have an equality and therefore this state survives the GSO projection.

This is the statement that the graviton multiplet is always in the spectrum of a string theory. This is also true in the bosonic and the type II string theories, which has the states (respectively)

$$\begin{aligned} &\partial X^{\nu}_{1}\;\partial\bar{X}^{\mu}_{1}\,|0\rangle_{NS} \\ &\psi^{\nu}_{1/2}\;\psi^{\mu}_{1/2}\,|0\rangle_{NS} \end{aligned}$$

which also always exists in the massless spectrum. It is seen that gravity is always present.

Continuing with the projection of the other states

$$e^{i\pi\mathbb{1}\cdot F_{NS}}: \quad \begin{matrix} \psi^\mu & \bar{\phi}^a & \bar{\phi}^b \\ -1 & -1 & -1 \end{matrix}\;|0\rangle = \delta_1 = -1 \qquad \checkmark \tag{4.15a}$$

The exponential on the left hand side can be explicitly written by

$$e^{(i\pi(\mathbb{1}(\psi^\mu)\cdot F_{NS}+\mathbb{1}(\bar{\phi}^a)\cdot F_{NS}+\mathbb{1}(\bar{\phi}^b))\cdot F_{NS}} = e^{i\pi(\mathbb{1}(\psi^\mu)\cdot F_{NS})}e^{i\pi(\mathbb{1}(\bar{\phi}^a))\cdot F_{NS})}e^{i\pi(\mathbb{1}(\bar{\phi}^b))\cdot F_{NS})}$$

and as these states are periodic in the basis vector $\mathbb{1}$ and $F_{NS} = +1$ then the contribution of each exponential is -1.

The same reasoning applies to the following states

$$e^{i\pi \mathbb{1}\cdot F_{NS}} : \begin{array}{ccccccc} \{\chi & y & w\} & \bar{\phi}^a & \bar{\phi}^b & |0\rangle_{NS} \\ & -1 & & -1 & -1 & \end{array} = \delta_1 = -1 \qquad \checkmark \tag{4.15b}$$

$$e^{i\pi \mathbb{1}\cdot F_{NS}} : \begin{array}{cccccc} \{\chi & y & w\} & \partial \bar{X}_1^\mu & |0\rangle_{NS} \\ & -1 & & 0 & \end{array} = \delta_1 = -1 \qquad \checkmark \tag{4.15c}$$

The boson state $\partial \bar{X}_1^\mu$ has a contribution of zero as bosons are not included in the fermion number $F(f)$ and therefore give no contribution.
As all of these states equal -1 on the left and right hand sides meaning there is an equality and they all survive the GSO projection.
Now we must consider the tachyons

$$e^{i\pi \mathbb{1}\cdot F_{NS}} : \begin{array}{cc} \bar{\phi}^a_{1/2} & |0\rangle_{NS} \\ -1 & \end{array} = \delta_1 = -1 \qquad \checkmark \tag{4.16}$$

Therefore the tachyon survives the GSO projection in this model.

4.1.4 Modular Invariance of $B = \mathbb{1}$ Model

The partition function for this model is composed of three sectors

$$Z = C\begin{pmatrix}0\\0\end{pmatrix} Z\begin{pmatrix}0\\0\end{pmatrix} + C\begin{pmatrix}0\\1\end{pmatrix} Z\begin{pmatrix}0\\1\end{pmatrix} + C\begin{pmatrix}1\\0\end{pmatrix} Z\begin{pmatrix}1\\0\end{pmatrix} \tag{4.17}$$

By using rule (3) of the one loop phases (equation (3.9)) we find the following to be true

$$C\begin{pmatrix}1\\0\end{pmatrix} = e^{i\frac{\pi 0\cdot 1}{2}} C\begin{pmatrix}0\\1\end{pmatrix}^* = \delta_1 = -1 \tag{4.18}$$

and using the result of equations (4.10) and (4.12) we calculate our partition function to be

$$Z = Z\begin{pmatrix}0\\0\end{pmatrix} - Z\begin{pmatrix}0\\1\end{pmatrix} - Z\begin{pmatrix}1\\0\end{pmatrix} \tag{4.19}$$

We now have to ask if this modular invariant. By using our definition of the partition function given in equation (2.80) we obtain

$$Z\begin{pmatrix}0\\0\end{pmatrix} = \frac{\vartheta_3^{1/2}}{\eta^{1/2}} = \frac{\vartheta_3^{20/2}}{\eta^{20/2}} \cdot \frac{\bar{\vartheta}_3^{44/2}}{\eta^{44/2}} = \frac{\vartheta_3^{10}}{\eta^{10}} \frac{\bar{\vartheta}_3^{22}}{\bar{\eta}^{22}} \tag{4.20}$$

The ϑ are the left movers and the power of 10 comes from the fact that there are 20 left moving fermions. The $\bar{\vartheta}$ are the right movers and the power of 22 comes from the fact that there are 44 right moving fermions.
Similarly for the other spin structures

$$Z\begin{pmatrix}0\\1\end{pmatrix} = \frac{\vartheta_4^{10}}{\eta^{10}} \frac{\bar{\vartheta}_4^{22}}{\bar{\eta}^{22}} \tag{4.21a}$$

$$Z\begin{pmatrix}1\\0\end{pmatrix} = \frac{\vartheta_2^{10}}{\eta^{10}} \frac{\bar{\vartheta}_2^{22}}{\bar{\eta}^{22}} \tag{4.21b}$$

Substituting these expressions into the total partition function given in equation (4.19) we get

$$Z = \frac{1}{\eta^{10}\bar{\eta}^{22}}\left[\vartheta_3^{10}\bar{\vartheta}_3^{22} - \vartheta_4^{10}\bar{\vartheta}_4^{22} - \vartheta_2^{10}\bar{\vartheta}_2^{22}\right] \tag{4.22}$$

Now we can consider the modular invariance of the partition function.
Under the transformation $\tau \to \tau + 1$ the following transformations are true

$$\eta \to e^{i\pi/12}\eta \tag{4.23a}$$

$$\vartheta_2 \to e^{i\pi/4}\vartheta_2 \tag{4.23b}$$

$$\vartheta_3 \leftrightarrow \vartheta_4 \tag{4.23c}$$

By performing these transformations on the partition function given in equation (4.22) we obtain

$$\begin{aligned} \to \quad & \frac{1}{e^{i\pi(10-22)/12}\eta^{10}\bar{\eta}^{22}}\left[\vartheta_4^{10}\bar{\vartheta}_4^{22} - \vartheta_3^{10}\bar{\vartheta}_3^{22} - e^{i\pi\frac{10-22}{4}}\vartheta_2^{10}\bar{\vartheta}_2^{22}\right] \\ = \; & \frac{1}{e^{-i\pi}\eta^{10}\bar{\eta}^{22}}\left[\vartheta_4^{10}\bar{\vartheta}_4^{22} - \vartheta_3^{10}\bar{\vartheta}_3^{22} - e^{-3i\pi}\vartheta_2^{10}\bar{\vartheta}_2^{22}\right] \end{aligned} \tag{4.24}$$

We then compare the *modular weight* between the initial and final partition functions to see if the transformation has left it invariant. In equation (4.22) the three terms had the weight $(+,-,-)$ and therefore was $+$ overall. After the transformation, the modular weight for each term is $-1 \times (+,-,+) = (-,+,-)$ and is therefore still $+$ overall. This means it is invariant.

Under the transformation $\tau \to -\frac{1}{\tau}$ we get

$$\eta \to (-i\tau)^{\frac{1}{2}}\,\eta \tag{4.25a}$$

$$\frac{\vartheta_2}{\eta} \leftrightarrow \frac{\vartheta_4}{\eta} \tag{4.25b}$$

$$\frac{\vartheta_3}{\eta} \to \frac{\vartheta_3}{\eta} \tag{4.25c}$$

so trivially, it is invariant under this transform.
The modular invariance of the partition function was expected, but it is a good check to know what is going on.

As we saw, this model contains a tachyon. As we know, a tachyon is unphysical which means that our constructed model is unstable. We will attempt to find a more accurate model by adding another basis vector b_2 which is what we will consider in the next section.

4.2 Basis Vector Model $B = \{\mathbb{1}, \vec{S}\}$

We begin by adding another basis vector $b_2 = \vec{S}$ which means we have $B = \{\mathbb{1}, \vec{S}\}$. The newly introduced basis vector has the following boundary conditions for the fermions

$$\vec{S}: \qquad S\{\psi^{\mu}_{1,2}\ ,\ \chi_{1,2}\ ,\ \chi_{3,4}\ ,\ \chi_{5,6}\} = 1 \qquad \to \text{Periodic (Ramond)} \tag{4.26}$$

and all other fermions are anti-periodic (Neuvu-Schwarz).

As an aside, we will find that the χ fermions are the Majorana-Weyl superpartners to the six compactified dimensions (degrees of freedom of the world-sheet). It is this reason that they are included in the $\vec{S}$ sector.

This basis vector must also satisfy the ABK rules

Rules Check:

First we will calculate $\vec{S}\cdot\vec{S}$ and $\vec{S}\cdot\vec{\mathbb{1}}$ as these will be used in the calculations. Using equation (3.6) we find

$$\vec{S}\cdot\vec{S} = \frac{8}{2} - 0 = 4 - 0 = 4 \tag{4.27a}$$

$$\vec{S}\cdot\vec{\mathbb{1}} = 4 - 0 = 4 \tag{4.27b}$$

These are true because we have 8 left moving real fermions (by definition in equation (2.50) and (4.26)).

As $N_S = 2$, rule 3 is satisfied by

$$\begin{aligned} N_S \cdot b_S \cdot b_S &= N_S \cdot \vec{S}\cdot\vec{S} \\ &= 2\cdot 4 \\ &= 8 \\ &= 0 \mod 8 \quad \checkmark \end{aligned} \tag{4.28}$$

and rule 2 is satisfied by

$$\begin{aligned} N_{S,\mathbb{1}} \cdot b_S \cdot b_{\mathbb{1}} &= N_{S,\mathbb{1}} \cdot \vec{S}\cdot\vec{\mathbb{1}} \\ &= 1\cdot 4 \\ &= 4 \\ &= 0 \mod 4 \quad \checkmark \end{aligned} \tag{4.29}$$

We have four sectors which are in the additive group Ξ

$$\{\mathbb{1}, S, \mathbb{1}+S, NS\} \tag{4.30}$$

The $\mathbb{1}+S$ state comes from the fact that we add together the S basis vector with the other basis vectors, in this case the only other basis vector that gives a contribution is $\mathbb{1}$ (as $S+NS$ does not give a new basis vector due to the other basis vector being NS). We will see this clearer later when constructing the NAHE set which requires adding more complex basis vectors than just $\mathbb{1}$.

The sectors $\{\mathbb{1}, \mathbb{1}+S\}$ do not give massless states, which can be seen by considering the mass squared matching condition (equation (3.12)). Firstly, for the $(\mathbb{1}+S)_L$ sector

$$\begin{aligned} \alpha_L\cdot\alpha_L &= (\mathbb{1}+S)_L\cdot(\mathbb{1}+S)_L = 6 \\ \Rightarrow M_L^2 &= -\frac{1}{2} + \frac{6}{8} = +\frac{1}{4} > 0 \end{aligned} \tag{4.31}$$

where the factor of 6 comes from the 12 left moving fermions $\frac{y_{1,\ldots,6}w_{1,\ldots,6}}{2} = 6$. For the $(1+S)_R$ sector

$$\begin{aligned} \alpha_R\cdot\alpha_R &= (\mathbb{1}+S)_R\cdot(\mathbb{1}+S)_R = 22 \\ \Rightarrow M_R^2 &= -1 + \frac{22}{8} > 0 \end{aligned} \tag{4.32}$$

where the value of 22 comes from the 44 real right moving fermions and $\frac{44}{2} = 22$. We should note that in the descriptions of the mass squared, there are no fermionic oscillators. This means this sector is a purely Ramond vacua.

The NS sector gives the same states as in the $B = \{\mathbb{1}\}$ model, but we have to perform the GSO projections of the S sector.

4.2.1 GSO Projections of the S Sector

For a general state s, we know from equation (3.16) that the state must satisfy

$$\begin{aligned} e^{i\pi\vec{S}\cdot\vec{F}_{NS}} \left|s\right\rangle_{NS} &= \delta_{NS}\, C\begin{pmatrix} NS \\ S \end{pmatrix} \left|s\right\rangle_{NS} \\ &= \delta_{NS}\delta_S \left|s\right\rangle_S \\ &= \delta_S \left|s\right\rangle_{NS} \\ &= -\left|s\right\rangle_{NS} \end{aligned} \tag{4.33}$$

where we used equation (4.12) to find $C\binom{NS}{S} = \delta_S$ and the result found earlier $\delta_{NS} = +1$. We find that $\delta_S = -1$ by using the definition in equation (4.26) that the boundary conditions of the fermions in S are Ramond (periodic) and therefore equal 1, using equation (3.11) for the spacetime spin index then gives the result $\delta_S = -1$.
Now we can consider the specific states in this model.

- For our 'graviton' state we find

$$\begin{aligned} e^{i\pi(1-0)}\,\{\psi^{\mu}_{1/2}\,\partial X^{\nu}_{+1}\left|0\right\rangle_{NS}\} &= \delta_S\{\psi^{\mu}_{1/2}\,\partial_{+1}X^{\nu}\left|0\right\rangle_{NS}\} \\ &= -\{\psi^{\mu}_{1/2}\,\partial_{+1}X^{\nu}\left|0\right\rangle_{NS}\} \end{aligned} \tag{4.34}$$

  where the 1 in the arguement of the exponential comes from the fact that $\psi^{\mu}_{1/2}$ is a fermion and the 0 comes from the fact that ∂X^{ν}_{+1} is a boson. Therefore $e^{i\pi} = -1$ and the contribution from the exponential is -1 . Similarly for the boson, we find the contribution of the exponential is $+1$ due to $e^0 = 1$.

  When we combine these two results (by multiplication), we find that the state is negative overall and therefore survives the GSO projection (as was expected because this state always exists in string theories).

- Now the second state

$$\begin{aligned} & e^{i\pi S\cdot F_{NS}}\left(\{\chi_{1/2}\ ;\ y_{1/2}\ ;\ w_{1/2}\}\,\partial\bar{X}^{\mu}_{+1}\left|0\right\rangle_{NS}\right) \\ & = e^{i\pi(S(\chi)\cdot F_{NS}+S(y)\cdot F_{NS}+S(w)\cdot F_{NS}+S(\partial\bar{X})\cdot F_{NS})}\left(\{\chi_{1/2}\ ;\ y_{1/2}\ ;\ w_{1/2}\}\,\partial\bar{X}^{\mu}_{+1}\left|0\right\rangle_{NS}\right) \end{aligned} \tag{4.35}$$

  As $F_{NS} = 1$ for all the fermions and 0 for the boson, we find

$$= e^{i\pi(S(\chi)+S(y)+S(w)+0)}\left(\{\chi_{1/2}\ ;\ y_{1/2}\ ;\ w_{1/2}\}\,\partial\bar{X}^{\mu}_{+1}\left|0\right\rangle_{NS}\right)$$

By definition, y and w are not in the $\vec{S}$ sector and therefore $S(y)$ and $S(w)$ equal zero as they are not periodic, we therefore get

$$= e^{i\pi(1+0+0+0)}\left(\{\chi_{1/2}\ ;\ y_{1/2}\ ;\ w_{1/2}\}\ \partial\bar{X}^{\mu}_{+1}\,|0\rangle_{NS}\right)$$

which leaves us with the result

$$\begin{array}{lcccccl} & \{\chi_{1/2}; & y_{1/2}; & w_{1/2}\} & \partial\bar{X}^{\mu}_{+1} & |0\rangle_{NS} & = -1 \\ e^{i\pi S}: & -1 & +1 & +1 & +1 & & = -1 \end{array} \tag{4.36}$$

The fermions $\chi_{1/2}$ have a negative result for the exponential. When this is multipied by the positive contribution from the boson an overall negative result remains, meaning this state $\chi_{1/2}\,\partial\bar{X}^{\mu}_{+1}\,|0\rangle_{NS}$ survives.
However, the other two fermions y, w have a positive contribution when individually multiplied by the positive contribution of the boson, this gives an overall positive result and the states $y_{1/2}\ \partial\bar{X}^{\mu}_{+1}\,|0\rangle_{NS}$ and $w_{1/2}\ \partial\bar{X}^{\mu}_{+1}\,|0\rangle_{NS}$ are projected out.

If we recall that this state corresponds to the gauge bosons of $SU(2)^6$ in the adjoint representation, then we can see that this projection breaks this gauge group as the fermions $\{\chi, y, w\}$ have been split due to the addition of this basis vector S.

- Following a similar method to the second state, for the third state we have our exponential

$$e^{i\pi\left(S(\psi^{\mu})\cdot F_{NS}(\psi^{\mu})+S(\bar{\phi}^{a})\cdot F(\phi^{a})+S(\bar{\phi}^{b})\cdot F(\phi^{b})\right)} = e^{i\pi(1+0+0)} \tag{4.37}$$

which leads to

$$\begin{array}{lcccccc} & \psi^{\mu}_{1/2} & \bar{\phi}^{a}_{1/2} & \bar{\phi}^{b}_{1/2} & |0\rangle_{NS} & & \\ e^{i\pi S\cdot F_{NS}}: & -1 & +1 & +1 & & = -1 & \checkmark \end{array} \tag{4.38}$$

Here the state is not projected as $-1 \times +1 \times +1 = -1$, this means the adjoint representation of SO(44) survives.

- The fourth state projection is (corresponding to the scalars in the adjoint of $SO(44)$)

$$\begin{array}{cccccc} \{\chi_{1/2}; & y_{1/2}; & w_{1/2}\} & \bar{\phi}^{a}_{1/2} & \bar{\phi}^{b}_{1/2} & |0\rangle_{NS} \\ \{-1; & +1; & +1\} & +1 & +1 & \\ \checkmark & \times & \times & & & \end{array} \tag{4.39}$$

Again, as the contributions for y and w are $+1 \times +1 \times +1 = +1$ which do not satisfy the GSO projection, they are projected out while the χ fermions remain.

- We also had the tachyon

$$e^{i\pi(0\cdot 1)}\{\bar{\phi}^{a}_{1/2}\,|0\rangle_{NS}\} = +\{\bar{\phi}^{a}_{1/2}\,|0\rangle_{NS}\} \tag{4.40}$$

As the GSO equaltiy is not satisfied due to the positive result, this means the tachyon is projected! This is a good sign.

4.2.2 S Sector and the GSO Projections

We have considered the NS sector and the $1+S$ sector so we now move on to consider the $\vec{S}$ sector. We will do this by first defining some properties of the sector, such as the mass squared and the vacuum, then we will consider the GSO projections of this sector.
We begin with $\alpha_L = S_L$

$$\begin{aligned} S_L \cdot S_L &= 4 \\ \Rightarrow M_L^2 &= -\frac{1}{2} + \frac{4}{8} = 0 \end{aligned} \tag{4.41a}$$

The 4 comes from the fact that there are 8 real moving left fermions in the $\vec{S}_L$ sector, as defined in equation (4.26).
Again, there are no oscillators as this is a purely Ramond vacua. There are no right moving fermions in the $\vec{S}_R$ sector, therefore

$$\begin{aligned} S_R \cdot S_R &= 0 \\ \Rightarrow M_R^2 &= -1 + \frac{0}{8} = -1 \end{aligned} \tag{4.41b}$$

This equation has a negative mass squared. This implies the need for either two right moving fermionic oscillators or one right moving bosonic oscillator in order to not have a tachyonic state.

It will be useful for later discussion to *complexify* our fermions. We do this by defining

$$\psi_{12}^{\mu} = \frac{1}{\sqrt{2}}(\psi_1^{\mu} + i\psi_2^{\mu}) \tag{4.42a}$$

$$\chi_{12} = \frac{1}{\sqrt{2}}(\chi_1 + i\chi_2) \tag{4.42b}$$

$$\chi_{34} = \frac{1}{\sqrt{2}}(\chi_3 + i\chi_4) \tag{4.42c}$$

$$\chi_{56} = \frac{1}{\sqrt{2}}(\chi_5 + i\chi_6) \tag{4.42d}$$

along with their complex conjugates given generally by $\lambda_{ab}^* = \frac{1}{\sqrt{2}}(\lambda_a - i\lambda_b)$. We will begin using these definitions now.

The S-vacuum is constructed as

$$|s\rangle_L = \underset{\psi^{\mu}}{|\pm\rangle} \quad \underset{\chi_{12}}{|\pm\rangle} \quad \underset{\chi_{34}}{|\pm\rangle} \quad \underset{\chi_{56}}{|\pm\rangle} \quad |0\rangle_L \tag{4.43}$$

So there are $2^4 = 16$ possible choices of state in this sector.

The states in the S-vacuum are

$$|s\rangle_L \ \partial\bar{X}^{\mu}_{+1} \, |0\rangle_R \tag{4.44a}$$

$$|s\rangle_L \ \bar{\phi}^a_{1/2}\bar{\phi}^b_{1/2} \ |0\rangle_R \tag{4.44b}$$

which is due to our need to have either one right moving bosonic oscillator or two right moving fermionic oscillators so as not to introduce a tachyonic state as mentioned above.

We should note that the states in the left-moving sector are in the Ramond vacuum of the spacetime fermions and therefore correspond to the fermions defined as periodic in the basis vector S as given in equation (4.26).

We now make some comments on these states. The states in equation (4.43) are in the spinorial (spin $1/2$) representation of the Lorentz group.
The states $|s\rangle_L\, \partial\bar{X}^{\mu}_{-1}\,|0\rangle_R$ have spin $3/2$ and are interpreted as gravitinos, whereas the states $|s\rangle_L\, \bar{\phi}^a_{1/2}\bar{\phi}^b_{1/2}\,|0\rangle_R$ are spin $1/2$.
We still have to apply the GSO projections, which is what we will do now.

<u>GSO Projections</u>
Here we will introduce a combinatorial notation. In the example of the states

$$\begin{array}{ccccc} S: & \psi^{\mu} & \chi_{12} & \chi_{34} & \chi_{56} \\ & |\pm\rangle & |\pm\rangle & |\pm\rangle & |\pm\rangle \end{array} \tag{4.45}$$

we can introduce the notation

$$\left[\binom{4}{0}+\binom{4}{1}+\binom{4}{2}+\binom{4}{3}+\binom{4}{4}\right] \tag{4.46}$$

to describe the states, where $\binom{4}{i}$ counts the number of negative states $|-\rangle$, *i.e*

$$\begin{aligned} \binom{4}{0} &= |+\rangle\,|+\rangle\,|+\rangle\,|+\rangle \\ \binom{4}{2} &= |-\rangle\,|-\rangle\,|+\rangle\,|+\rangle \qquad etc \end{aligned} \tag{4.47}$$

We should also recall the fermion number

$$F:|\pm\rangle = \begin{cases} 0 & |+\rangle \\ -1 & |-\rangle \end{cases} \tag{4.48}$$

for the coming projections.

We have the GSO projections of two vectors $\{\mathbb{1}, S\}$ to perform. Firstly, we will consider the vector $\mathbb{1}$ and secondly the vector $\vec{S}$.
For the vector $\mathbb{1}$ we have the GSO projection

$$e^{i\pi\mathbb{1}\cdot F_S}\,|s\rangle_S = \delta_S\, C\binom{S}{\mathbb{1}}^{*}|s\rangle_S \tag{4.49}$$

By using rule (1) for the one loop phases (equation (3.7)) we find a result for the one loop phase in the above equation

$$C\binom{b_i}{b_j} = C\binom{S}{\mathbb{1}} = \delta_S e^{\frac{2\pi i n_1}{N_1}} = \delta_{\mathbb{1}} e^{\frac{2\pi i m_S}{N_S}}\, e^{\frac{i\pi}{2} b_S\cdot b_{\mathbb{1}}} \tag{4.50}$$

If we consider the contribution of this equation to the GSO projection, we find that two of the exponentials contribute ± 1 by

$$C\binom{S}{\mathbb{1}} = \begin{array}{ccccccc} \delta_{\mathbb{1}} & e^{i\frac{\pi\mathbb{1}\cdot S}{2}} & e^{i\frac{2\pi m_S}{N_S}} & = & \delta_S & e^{i\frac{2\pi n_1}{N_1}} \\ (-1) & (+1) & (\pm 1) & = & (-1) & (\pm 1) \end{array} \tag{4.51}$$

(where $1 \cdot S = 4 - 0 = 4$ and therefore gives a positive contribution) and $m_S, n_1 = 0, 1$ due to the ABK rule given in equation (3.2) *i.e* $m_S, n_1 = 0$ mod 2 where $N_i = 2$. We therefore have two choices that we can make for the result of the GSO projection.
We will first consider the case where we make the choice

$$C \begin{pmatrix} S \\ \mathbb{1} \end{pmatrix} = -1 \tag{4.52}$$

Due to this choice, we also calculate

$$C \begin{pmatrix} \mathbb{1} \\ S \end{pmatrix} = e^{i\frac{\pi \mathbb{1}\cdot S}{2}} C \begin{pmatrix} S \\ \mathbb{1} \end{pmatrix}^* = -1 \tag{4.53}$$

We can also calculate the one loop phase of two S sectors. Using rule (2) of the one loop phases gives

$$C \begin{pmatrix} S \\ S \end{pmatrix} = -e^{i\frac{\pi S\cdot S}{4}} C \begin{pmatrix} S \\ \mathbb{1} \end{pmatrix}$$

substituting the result of equation (4.51)

$$C \begin{pmatrix} S \\ S \end{pmatrix} = -e^{i\frac{\pi S\cdot S}{4}} \ \delta_S \ e^{\frac{2i\pi n_1}{N_1}}$$

As $S \cdot S = 4$

$$C \begin{pmatrix} S \\ S \end{pmatrix} = -e^{i\pi} \ \delta_S \ e^{\frac{2i\pi n_1}{N_1}}$$

So the first exponential gives a negative contribution and the two other terms also give negative contributions (as in equation (4.51) because we make the choice in equation (4.52)). Therefore

$$C \begin{pmatrix} S \\ S \end{pmatrix} = (-)(-)(-) = -1 \tag{4.54}$$

Therefore, we have

$$\begin{array}{cc} & \begin{array}{cc} \mathbb{1} & S \end{array} \\ \begin{array}{c} \mathbb{1} \\ S \end{array} & \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \end{array} \quad \text{or} \quad \begin{array}{cc} & \begin{array}{cc} \mathbb{1} & S \end{array} \\ \begin{array}{c} \mathbb{1} \\ S \end{array} & \begin{pmatrix} -1 & +1 \\ +1 & +1 \end{pmatrix} \end{array} \tag{4.55}$$

Where the $\mathbb{1}$ and S are the basis vectors and inside the matrix there are the relative contributions.

Only $C\binom{S}{\mathbb{1}}$ or $C\binom{\mathbb{1}}{S}$ are independent, the rest are fixed by modular invariance. This becomes important when the models become more complicated.

We therefore have

$$e^{i\pi \mathbb{1}\cdot F_S} \left|s\right\rangle_S = + \left|s\right\rangle_S \tag{4.56}$$

$$\begin{aligned} e^{i\pi \mathbb{1}\cdot F_S} \left|s\right\rangle_L \bar{\phi}^a \bar{\phi}^b \left|0\right\rangle_R &= e^{-i\pi(\mathbb{1}\cdot(\#\text{of}|-\rangle)-(1\cdot1)-(1\cdot1))} \left|s\right\rangle_S \\ &= e^{i\pi(\mathbb{1}\cdot\#\text{of}|-\rangle)} \left|s\right\rangle \end{aligned} \tag{4.57}$$

If the number of $|-\rangle$ is odd, then the state is projected. If the number of $|-\rangle$ is even then the state survives.

So we have that only even states remain, or expressed using our combinatorial notation

$$|s\rangle_S = \left[\binom{4}{0} + \binom{4}{2} + \binom{4}{4}\right] \bar{\phi}^a_{1/2}\bar{\phi}^b_{1/2}|0\rangle \tag{4.58}$$

which is our state that has a spin 1/2 and corresponds to the gaugino.
The same is true for

$$\left[\binom{4}{0} + \binom{4}{2} + \binom{4}{4}\right] \partial\bar{X}^\mu_{1/2}|0\rangle \tag{4.59}$$

which is the state with spin 3/2 and corresponds to the gravitino. It is useful to ask how many gravitinos are present. We can find this by splitting the states to

$$\binom{1}{0}\left[\binom{3}{0} + \binom{3}{2}\right] + \binom{1}{1}\left[\binom{3}{1} + \binom{3}{3}\right] \tag{4.60}$$

where the $\binom{1}{0}$ and $\binom{1}{1}$ are the ψ^μ fermions and $\binom{3}{even/odd}$ correspond to $\chi_{12}, \chi_{34}, \chi_{56}$. These two terms corresponding to the χ fermions are the two components of a spacetime Weyl spinor.
Having 4 gravitinos present means that we have $\mathcal{N} = 4$ SUSY.

We still have to check the S projection which is what we will consider now.

$$e^{i\pi S\cdot F_S}\,|s\rangle_S \quad = \quad \delta_S \binom{S}{S}^* |s\rangle_S \quad = \quad +|s\rangle_S \tag{4.61}$$

We find that this projection actually acts in the same way as for the $\mathbb{1}$ vector because $S_R \equiv 0$. Therefore, the spectrum remains intact after the S projection. This means the model contains $\mathcal{N} = 4$ SUSY and a gauge group of $SO(44)$. This concludes the GSO projections of this model.

4.2.3 Partition Function of the $B = \{\mathbb{1}, \vec{S}\}$ Model

We now want to consider the partition function of the model which has the sectors $\{\mathbb{1}, S, \mathbb{1}+S, NS\}$.

Firstly, we recall that

$$\binom{1}{1} \rightarrow \vartheta_1(\tau) \equiv 0 \tag{4.62}$$

Therefore, whenever we have two intersecting periodic fermions, this term vanishes in the partition function.
For example

$$\binom{\mathbb{1}}{S};\binom{\mathbb{1}}{\mathbb{1}};\binom{\mathbb{1}+S}{\mathbb{1}} \equiv 0 \tag{4.63}$$

but

$$\binom{\mathbb{1}+S}{S} \neq 0 \tag{4.64}$$

The sectors that are non-zero are then

$$\binom{NS}{S}\binom{S}{NS}\binom{NS}{NS} \quad \binom{\mathbb{1}}{NS}\binom{\mathbb{1}+S}{NS}\binom{\mathbb{1}+S}{S} \quad \binom{NS}{\mathbb{1}}\binom{NS}{\mathbb{1}+S}\binom{S}{\mathbb{1}+S} \tag{4.65}$$

meaning there are nine non-zero sectors.

Recall $\begin{pmatrix}0\\0\end{pmatrix} \to \vartheta_3$, $\begin{pmatrix}0\\1\end{pmatrix} \to \theta_4, \begin{pmatrix}1\\0\end{pmatrix} \to \theta_2$.
So each sector can be written in the following way

$$\begin{aligned}
&-\begin{pmatrix}NS\\S\end{pmatrix} \to \theta_4^4\,\theta_3^6\,\bar\theta_3^{22}\\
&-\begin{pmatrix}S\\NS\end{pmatrix} \to \theta_2^4\,\theta_3^6\,\bar\theta_3^{22}\\
&+\begin{pmatrix}NS\\NS\end{pmatrix} \to \theta_3^4\,\theta_3^6\,\bar\theta_3^{22}\\
&-\begin{pmatrix}\mathbb{1}\\NS\end{pmatrix} \to \theta_2^4\,\theta_2^6\,\bar\theta_2^{22}\\
&+\begin{pmatrix}\mathbb{1}+S\\NS\end{pmatrix} \to \theta_3^4\,\theta_2^6\,\bar\theta_2^{22}\\
&-\begin{pmatrix}\mathbb{1}+S\\S\end{pmatrix} \to \theta_4^4\,\theta_2^6\,\bar\theta_2^{22}\\
&-\begin{pmatrix}NS\\\mathbb{1}\end{pmatrix} \to \theta_4^4\,\theta_4^6\,\bar\theta_4^{22}\\
&+\begin{pmatrix}NS\\\mathbb{1}+S\end{pmatrix} \to \theta_3^4\,\theta_4^6\,\bar\theta_4^{22}\\
&-\begin{pmatrix}S\\\mathbb{1}+S\end{pmatrix} \to \theta_2^4\,\theta_4^6\,\bar\theta_4^{22}
\end{aligned} \tag{4.66}$$

$$C\begin{pmatrix}NS\\S\end{pmatrix} = \delta_S = -1 \qquad\qquad C\begin{pmatrix}S\\NS\end{pmatrix} e^{i\pi\frac{NS\cdot S}{2}} C\begin{pmatrix}NS\\S\end{pmatrix}^* = -1 \tag{4.67}$$

$$C\begin{pmatrix}NS\\b_j\end{pmatrix} = C\begin{pmatrix}b_j\\NS\end{pmatrix} \tag{4.68}$$

$$C\begin{pmatrix}S\\\mathbb{1}+S\end{pmatrix} = \delta_S C\begin{pmatrix}S\\\mathbb{1}\end{pmatrix} C\begin{pmatrix}S\\S\end{pmatrix} = - \; - \; - = (-1) \tag{4.69}$$

$$C\begin{pmatrix}\mathbb{1}+S\\\mathbb{1}\end{pmatrix} = e^{\frac{[\mathbb{1}+S]\cdot[S]}{2}} C\begin{pmatrix}S\\\mathbb{1}+S\end{pmatrix}^* = C\begin{pmatrix}S\\\mathbb{1}+S\end{pmatrix} = -1 \tag{4.70}$$

$$Z = \left[\theta_3^4 - \theta_2^4 - \theta_4^4\right]\left[\theta_3^6\,\bar\theta_3^{22} + \theta_2^6\,\bar\theta_2^{22} + \theta_4^6\,\bar\theta_4^{22}\right] \tag{4.71}$$

This is the partition function of the $\mathcal{N}=4$ model.

$$\left[\theta_3^4 - \theta_2^4 - \theta_4^4\right] \equiv 0 \qquad \leftrightarrow \qquad \text{Jacobi Identity} \tag{4.72}$$

$$\Rightarrow \qquad \Lambda_{\text{Cosmological}} = 0 \tag{4.73}$$

4.2.4 Outlook

We have a model with no tachyons, but

1. The gauge group is $SO(44)$
2. There is no matter
3. We have $\mathcal{N} = 4$ SUSY

What we actually want a model to have is

1. Matter
2. Gauge group $\rightarrow$ Gauge Group $\subset SU(3) \times SU(2) \times U(1)$
3. $\mathcal{N} = 1$ SUSY

As the gauge group $SO(44)$ is too large, it implies that there are observable $\otimes$ hidden sectors. We need the matter to have three generations in order to retrieve the Standard Model and we want to have $\mathcal{N}$=1 SUSY as this gives us the most freedom in our models to recover phenomenologically correct results.

In order to (hopefully) solve some of these problems, we need to add more basis vectors. Each basis vector which is added gives rise to new sectors and at the same time imposes GSO projections on the previous sectors.

The process then goes as follows

1. Choose an intellegent basis vector (this is the creative part)
2. Check compatability with the rules
3. Check for massless states
4. Check the type of massless states, *i.e* the number of oscillators acting on the vacuum and whether they are purely Ramond
5. Analyze the Hilbert space from the new sectors $\{b_1; \ldots; b_n\}$ and impose the GSO projections on the states from these sectors
6. Impose the GSO of b_{n+1} on $H\{b_1, \ldots, b_n\}$

It is this which process we will consider in the next chapter.

Chapter 5

Constructing the NAHE Set

We will begin by adding new basis vectors which will lead us to form the commonly used NAHE set.

5.1 b_1 Vector

The first step of our process laid out at the end of the last chapter was to define an intellegent basis vector. The basis vector we add is labelled b_1 and is defined by

$$b_1 = \{\psi^\mu_{1,2},\ \chi_{12},\ y_3y_4,\ y_5y_6 \mid \overline{y_3y_4},\ \overline{y_5y_6},\ \bar{\psi}^\mu_{1,\ldots,5},\ \bar{\eta}_1\} \tag{5.1}$$

The basis of this model is then

$$B = \{\mathbb{1}, S, b_1\} \tag{5.2}$$

which means there are $2^3 = 8$ sectors, so the additive group is

$$\Xi = \{NS, \mathbb{1}+S, \mathbb{1}+b_1, S+b_1, \mathbb{1}+S+b_1, \mathbb{1}, S, b_1\} \tag{5.3}$$

The second step of our process was to check that the rules are satisfied. We check the ABK rules by:
Rule (2) is considered by

$$N_{1b_1}\ 1 \cdot b_1 = 2(4-8) = 0 \mod 4 \qquad \checkmark \tag{5.4a}$$

since there are 8 left moving real fermions and 8 right moving complex fermions.

$$N_{Sb_1} \cdot S \cdot b_1 = 2(2-0) = 0 \mod 4 \qquad \checkmark \tag{5.4b}$$

since there is an 'overlap' between 2 left moving real fermions and no right moving fermions the S and b_1 sector.
Rule (3) is considered by

$$N_{b_1} \cdot b_1 \cdot b_1 = 2(4-8) = 0 \mod 8 \qquad \checkmark \tag{5.4c}$$

where there are 4 left moving real fermions and 8 right moving complex fermions. Here $N_{b_1} = 2$.

The next step in our process is checking for massless states. First we calculate that for the left and right moving sectors respectively, we have

$$b_{1L} \cdot b_{1L} = 4 \qquad\qquad b_{1R} \cdot b_{1R} = 8 \tag{5.5}$$

which we can substitute into our equation for the mass squared as follows

$$M_L^2 = -\frac{1}{2} + \frac{4}{8} + 0 = -1 + \frac{8}{8} + 0 = M_R^2 \tag{5.6}$$

where there is a $+0$ as there are no oscillators present, meaning that the vacuum is purely Ramond. We can see that there are massless states present but no tachyonic states have been introduced by this choice of basis vector.

5.1.1 GSO Projections of b_1

The GSO projections from an analysis of the Hilbert space are the next step in our process. We begin by calculating the one loop GSO coefficients

$$C\binom{\alpha}{\beta} \qquad \rightarrow \qquad \begin{array}{c c} & \begin{array}{ccc} \mathbb{1} & S & b_1 \end{array} \\ \begin{array}{c} \mathbb{1} \\ S \\ b_1 \end{array} & \begin{pmatrix} -1 & -1 & \cdot \\ -1 & -1 & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \end{array} \tag{5.7}$$

We have two new independent GSO's: $C\binom{\mathbb{1}}{b_1}$ and $C\binom{S}{b_1}$, which we can calculate by using rule (1) of the one loop coefficients

$$C\binom{\mathbb{1}}{b_1} = \delta_{\mathbb{1}} e^{i\frac{2\pi m}{N_{b_1}}} = \delta_{b_1} e^{i\pi\frac{b_1\cdot\mathbb{1}}{2}} e^{i\frac{2\pi n}{N_{\mathbb{1}}}} = \pm 1 \qquad \text{choose } (-1) \tag{5.8a}$$

$$C\binom{S}{b_1} = \delta_{S} e^{i\frac{2\pi m}{N_{b_1}}} = \delta_{b_1} e^{i\pi\frac{S\cdot b_1}{2}} e^{i\frac{2\pi n}{N_{S}}} = \pm 1 \qquad \text{choose } (+1) \tag{5.8b}$$

Using rule (3)

$$C\binom{b_1}{\mathbb{1}} = e^{i\pi\frac{b_1\cdot\mathbb{1}}{2}}\, C\binom{\mathbb{1}}{b_1}^* = C\binom{\mathbb{1}}{b_1} = +1 \tag{5.8c}$$

$$C\binom{b_1}{S} = e^{i\pi\frac{b_1\cdot S}{2}}\, C\binom{S}{b_1}^* = (-1)(+1) = (-1) \tag{5.8d}$$

where equation (5.8d) was found to be negative as the exponential gives a negative contribution, which can also be seen in equation (5.8b).

First, lets anaylze the GSO of b_1 on the NS sector.
From the state $\{\chi^i_{1/2}, y^i_{1/2}, w^i_{1/2}\}\partial\bar{X}^\mu_1 |0\rangle_{NS}$ which is in the NS sector (equation (4.7)), we consider the χ fermions and perform the GSO projections by adhearing to equation (5.8a)

$$\begin{array}{cccc} & \chi^{1,2}_{1/2}\,\partial X^\mu_+\ |0\rangle_{NS} & \chi^{3,4}_{1/2}\,\partial X^\mu_+\ |0\rangle_{NS} & \chi^{5,6}_{1/2}\,\partial X^\mu_+\ |0\rangle_{NS} \\ e^{i\pi b_1 F_{NS}} & -1 & +1 & +1 \\ & \checkmark & \times & \times \end{array} \tag{5.9}$$

So we are only left with the $\chi^{1,2}_{1/2}\partial X^\mu_+ \, |0\rangle_{NS}$ state and the other two are projected out. This was expected as only the $\chi^{1,2}_{1/2}$ were defined to be in the basis vector b_1 in equation (5.1).
The GSO projection for the next state (given in equation (4.6)) is

$$\begin{array}{ccccc} & \psi^\mu & \bar{\phi}^a_{1/2} & \bar{\phi}^b_{1/2} & |0\rangle_{NS} \\ e^{i\pi b_1\cdot F_{NS}} & -1 & \{ & \} & \end{array} \tag{5.10}$$

where the curly brackets repesent the four possible choices of boundary conditions for the ϕ's

$$\begin{array}{ccccc} \bar{\phi}^a_{1/2}\bar{\phi}^b_{1/2}: & \{1,1\} & \{1,0\} & \{0,1\} & \{0,0\} \\ & -1-1 & -1+1 & +1-1 & +1+1 \end{array} \tag{5.11}$$

we find that the states $\{1,1\}$ and $\{0,0\}$ are the ones that survive.
For the explicit states, which have one left moving fermion and two right moving fermions by definition (equation (4.6)), our GSO projections are then

- For $\{1,1\}$

$$\begin{array}{ccccc} & \psi^\mu_{1/2} & \{\bar{\psi}^{1,\dots,5},\bar{\eta}_1,\bar{y}_{3,4},\bar{y}_{5,6}\} & \{\bar{\psi}^{1,\dots,5},\bar{\eta}_1,\bar{y}_{3,4},\bar{y}_{5,6}\} & |0\rangle_{NS} \\ e^{i\pi b_1\cdot F_{NS}} & -1 & -1 & -1 & \checkmark \end{array} \tag{5.12a}$$

- For $\{0,0\}$

$$\begin{array}{ccccc} & \psi^\mu_{1/2} & \{\bar{w}_{1,\dots,6},\bar{y}_{1,2},\bar{\eta}_2,\bar{\eta}_3,\bar{\phi}_{1,\dots,8}\} & \{\bar{w}_{1,\dots,6},\bar{y}_{1,2},\bar{\eta}_2,\bar{\eta}_3,\bar{\phi}_{1,\dots,8}\} & |0\rangle_{NS} \\ e^{i\pi b_1\cdot F_{NS}} & -1 & +1 & +1 & \checkmark \end{array} \tag{5.12b}$$

- For $\{0,1\}$ and $\{1,0\}$

$$\begin{array}{ccccc} & \psi^\mu_{1/2} & \{\bar{\psi}^{1,\dots,5},\bar{\eta}_1,\bar{y}_{3,4},\bar{y}_{5,6}\} & \{\bar{w}_{1,\dots,6},\bar{y}_{1,2},\bar{\eta}_2,\bar{\eta}_3,\bar{\phi}_{1,\dots,8}\} & |0\rangle_{NS} \\ e^{i\pi b_1\cdot F_{NS}} & -1 & -1 & +1 & \times \end{array} \tag{5.12c}$$

So we can see that the states in equation (5.12a) and (5.12b) survive, but the states in equation (5.12c) are projected.
As this is the case, we find that the gauge group $SO(44)$ breaks to $SO(16)\times SO(28)$. This can be seen as each bracket in equation (5.12a) contains 8 fermions and as the gauge group is $SO(2n)$ then we find that the gauge group is $SO(2\times 8) = SO(16)$. Similarly for the brackets in (5.12b), there are 14 fermions and therefore the resulting gauge group is $SO(2\times 14) = SO(28)$.

The last set of states to GSO project are those given in equation (4.8)

$$\begin{array}{ccccc} & \chi_{1,\dots,6} & \bar{\phi}^a_{1/2} & \bar{\phi}^b_{1/2} & |0\rangle_{NS} \\ e^{i\pi b_1 F_{NS}} & \pm1 & \{ & \} & \end{array} \tag{5.13}$$

This is done in a similar fashion to the states above. We get

- For $\{1,1\}$

$$\begin{array}{ccccc} & \chi_{1,2} & \{\bar\psi^{1,\dots,5},\bar\eta_1,\bar y_{3,4},\bar y_{5,6}\} & \{\bar\psi^{1,\dots,5},\bar\eta_1,\bar y_{3,4},\bar y_{5,6}\} & |0\rangle_{NS} \\ e^{i\pi b_1 F_{NS}} & -1 & -1 & -1 & \checkmark \end{array} \tag{5.14a}$$

- For $\{0,0\}$

$$\begin{array}{ccccc} & \chi_{1,2} & \{\bar w_{1,\dots,6},\bar y_{1,2},\bar\eta_2,\bar\eta_3,\bar\phi_{1,\dots,8}\} & \{\bar w_{1,\dots,6},\bar y_{1,2},\bar\eta_2,\bar\eta_3,\bar\phi_{1,\dots,8}\} & |0\rangle_{NS} \\ e^{i\pi b_1 F_{NS}} & -1 & +1 & +1 & \checkmark \end{array} \tag{5.14b}$$

- For $\{0,1\}$ and $\{1,0\}$

$$\begin{array}{ccccc} & \chi_{1,2} & \{\bar\psi^{1,\dots,5},\bar\eta_1,\bar y_{3,4},\bar y_{5,6}\} & \{\bar w_{1,\dots,6},\bar y_{1,2},\bar\eta_2,\bar\eta_3,\bar\phi_{1,\dots,8}\} & |0\rangle_{NS} \\ e^{i\pi b_1 F_{NS}} & -1 & -1 & +1 & \times \end{array} \tag{5.14c}$$

and therefore only the states $\{0,0\}$ and $\{1,1\}$ survive. We must also check for the states containing $\chi_{34,56}$ which give a positive contribution. We find that the states $\{1,0\},\{0,1\}$ survive and the states $\{0,0\},\{1,1\}$ are projected.

Next we need to analyze the GSO projections of b_1 on the S sector.

$$S: \quad \begin{array}{cccc} \{\psi^\mu_{1/2} & \chi_{1,2} & \chi_{3,4} & \chi_{5,6}\} \\ |\pm\rangle & |\pm\rangle & |\pm\rangle & |\pm\rangle \end{array} \quad = 1 \tag{5.15}$$

For our GSO projections we want to satisfy equation (5.8d) *i.e* we want a (-1).

For the gravitino state, which is

$$\left[\binom{4}{0}+\binom{4}{2}+\binom{4}{4}\right]\partial\bar X^\mu_{1/2}\,|0\rangle \tag{5.16}$$

in the basis vector b_1 we have

$$\begin{array}{cccccc} & \psi^\mu_{1/2} & \chi_{1,2} & \chi_{3,4} & \chi_{5,6} & |0\rangle \\ b_1 & 1 & 1 & 0 & 0 & \end{array} \tag{5.17}$$

as the first two fermions are defined to be within b_1. As we have two periodic and two anti-periodic boundary conditions, we only need to consider the $\binom{4}{2}$ states.

After the GSO projections, we find

$$\binom{4}{2}\xrightarrow{\text{breaks}}\left[\binom{2}{1}\right]\left[\binom{2}{1}\right]=\left[\binom{1}{0}\binom{1}{1}\right]\binom{2}{1}+\left[\binom{1}{1}\binom{1}{0}\right]\binom{2}{1} \tag{5.18}$$

and the $\binom{2}{1}$ can be identified to be describing 2 spin $3/2$ states. In terms of the fermions, the breaking that has occured is

$$\{\psi^\mu,\chi_{12},\chi_{34},\chi_{56}\}\to\{\psi^\mu,\chi_{12}\}\{\chi_{34},\chi_{56}\}$$

This shows that SUSY has broken from $\mathcal{N}=4$ to $\mathcal{N}=2$.

The next state we consider is

$$\left[\binom{4}{0}+\binom{4}{2}+\binom{4}{4}\right]\bar{\phi}^a_{1/2}\bar{\phi}^b_{1/2}\left|0\right\rangle_{Ramond} \tag{5.19}$$

Once again, we have that

$$\begin{array}{cccccc} & \psi^\mu_{1/2} & \chi_{1,2} & \chi_{3,4} & \chi_{5,6} & \left|0\right\rangle_{Ramond} \\ b_1 & 1 & 1 & 0 & 0 & \end{array} \tag{5.20}$$

When we perform the GSO projection and have $e^{i\pi b_1 \cdot F_S}$ acting on the left hand side of the state $\left|s\right\rangle_S$ (which is the combinatorial part) we find

$$\left[\binom{2}{0}+\binom{2}{2}\right]_+ \qquad \text{or} \qquad \left[\binom{2}{1}+\binom{2}{1}\right]_- \tag{5.21}$$

meaning we have either even states (denoted by a positive chirality) or odd states (denoted by a negative chirality). Now we consider the projections of the right hand side ($i.e$ on $\bar{\phi}^a_{1/2}\bar{\phi}^b_{1/2}$) we find

$$\begin{aligned} A: \quad & \left[\{\bar{\psi}^{1,\dots,5},\bar{\eta}_1,\bar{y}_{3,\dots,6}\}\{\bar{\psi}^{1,\dots,5},\bar{\eta}_1,\bar{y}_{3,\dots,6}\}\right]_+ \\ & \text{and } \{\bar{w}_{1,\dots,6},\bar{y}_{1,2},\bar{\eta}_2,\bar{\eta}_3,\bar{\phi}_{1,\dots,8}\}\{\bar{w}_{1,\dots,6},\bar{y}_{1,2},\bar{\eta}_2,\bar{\eta}_3,\bar{\phi}_{1,\dots,8}\}\Big]_+ \end{aligned} \tag{5.22a}$$

or

$$B: \left[\{\bar{\psi}^{1,\dots,5},\bar{\eta}_1,\bar{y}_{3,\dots,6}\}_-\{\bar{w}_{1,\dots,6},\bar{y}_{1,2},\bar{\eta}_2,\bar{\eta}_3,\bar{\phi}_{1,\dots,8}\}_+\right]_{\text{overall}=-} \tag{5.22b}$$

This means that the invariant states are

$$\left[\binom{2}{0}+\binom{2}{2}\right]\otimes B \qquad \text{and} \qquad \left[\binom{2}{1}+\binom{2}{1}\right]\otimes A \tag{5.23}$$

because we have matched the results that cancel the positive or negative signs on the brackets, therefore giving an overall neutral chirality state.

These produce the states that complete the $\mathcal{N}=2$ supersymmetric representiations. The sector S is the SUSY generator.

5.1.2 Remaining Supersymmetries

To know how many supersymmetries are left, it is adequate to calculate the number of gravitinos that are left in the massless spectrum. The state

$$\left|s\right\rangle_{S_{\text{left}}}\ \partial X^\mu\left|0\right\rangle_R \tag{5.24}$$

gives the gravitons of the model. In this state it is $\left|s\right\rangle_{S_{\text{left}}}$ which counts the number of gravitons.

If we have at least $\mathcal{N}=1$, it is enough to check the spectrum from $\{1,\dots,b_1,\dots b_n\}$ and then the sectors $S+\{1,\dots,b_1,\dots b_n\}$ will give the superpartners.

Two sectors $\vec{\alpha},\vec{\beta}\in\Xi$ which are related by $\vec{\alpha}=\vec{\beta}+S$ are related by the space time SUSY generator.

A convenient procedure which is easily implemented on a computer is to analyse all the sectors that give spacetime fermions $i.e$ $\alpha(\psi^\mu)=1$. Then all the sectors with $\alpha(\psi^\mu) = 0$ which give the spacetime scalar are related by the SUSY generators.

To know the massless spectrum content of a model it is therefore sufficient to analyse the massless fermions.

5.1.3 New States from b_1 Basis Vector

We have to find the new states from the sector b_1 that are left after our GSO projections. The states present from the definition of b_1 are

$$\begin{array}{lcccccccc} b_1: & \psi_{1,2} & \chi_{1,2} & y_3y_4 & y_5y_6 & \overline{y_3y_4} & \overline{y_5y_6} & \bar{\psi}_{1,\dots,5} & \bar{\eta}_1 \\ & \vert\pm\rangle & \vert\pm\rangle & \vert\pm\rangle & \vert\pm\rangle & \vert\pm\rangle & \vert\pm\rangle & \vert\pm\rangle & \vert\pm\rangle \\ & \Big[\binom{12}{0} & +\binom{12}{1} & +\dots & +\dots & +\dots & +\dots & +\binom{12}{11} & +\binom{12}{12}\Big] \end{array} \tag{5.25a}$$

and it is by performing the GSO projections with the other basis vectors present which determines which states are kept. For the GSO projection with the basis vector $\mathbb{1}$ and the choice $\delta_{b_1} C\binom{b_1}{\mathbb{1}} = +1$, we have the GSO equation

$$\begin{aligned} e^{i\pi \mathbb{1}\cdot F_{b_1}} \vert s\rangle_{b_1} &= \delta_{b_1} C\binom{b_1}{\mathbb{1}} \vert s\rangle_{b_1} \\ &= +\vert s\rangle_{b_1} \end{aligned}$$

This means our GSO projection leaves the states

$$\begin{array}{lcccccccc} \mathbb{1}: & 1 & 1 & 1 & 1 & 1 & 1 & (11111) & 1 \\ & \Big[\binom{12}{0} & +\binom{12}{2} & +\binom{12}{4} & +\dots & +\dots & +\dots & +\dots & +\binom{12}{12}\Big] \end{array} \tag{5.25b}$$

and therefore the odd states have been projected out. This is because in the exponential we need an even number of $\vert-\rangle$ states in order for the state to remain. Performing the GSO projections on the S sector where in the GSO projection $\delta_{b_1} C\binom{b_1}{S} = +1$ is chosen gives

$$\begin{array}{lccc} S: & 1 & 1 & 10\times 0's \\ & \Big[\binom{2}{0} & +\binom{2}{2}\Big] & \Big[\binom{10}{\text{even}}\Big] \end{array} \tag{5.25c}$$

and choosing $\delta_{b_1} C\binom{b_1}{b_1} = +1$ and performing the GSO projection on the b_1 sector gives

$$\begin{array}{lccc} b_1: & 1 & 1 & 10\times 1's \\ & \Big[\binom{2}{0} & +\binom{2}{2}\Big] & \Big[\binom{10}{\text{even}}\Big] \end{array} \tag{5.25d}$$

In equation (5.25a) there are 2^{12} states, in equation (5.25b) there are 2^{11} states and in equations (5.25 c,d) there are 2^{10} states.

These are the states which are left after the GSO projections. We can write these down as representations of the 4D gauge group, but this will not be particuarly illuminating currently.

As mentioned previously, we also have the $S + b_1$ sector

$$S + b_1 : \quad \chi_{3,4} \quad \chi_{5,6} \quad y_3y_4 \quad y_5y_6 \quad \overline{y_3y_4} \quad \overline{y_5y_6} \quad \bar{\psi}_{1,\ldots,5} \quad \bar{\eta}_1 \tag{5.26}$$

This gives the SUSY partners of b_1, which are spacetime bosons.

More basis vectors can be added in a similar manner to the above procedure. Upon introducing new vectors, there are new GSO projections on the previous spaces. New vectors also means we obtain new massless states.

However, in practice, since we have gained the experience by constructing previous models, we can speed this process up by first specifying the boundary conditions, then the GSO phases and analyse the spectrum in one go.

This is how we will continue by adding two more basis vectors b_2 and b_3.

We should remark that at this point the analysis of the partition function by hand becomes extremely tedious although not impossible. As we are mainly interested in the massless spectrum, the partition function is not particuarly useful to us. The massive spectrum only becomes useful for threshold corrections, among other things.

5.2 Contruction of the Full NAHE Set

The NAHE set has the basis vectors $\{\mathbb{1}, S, b_1, b_2, b_3\}$. The basis vectors b_2, b_3 are defined as

$$b_2 = \{\psi^\mu, \chi_{34}, y_{12}, w_{56} \mid \bar{y}_{12}, \bar{w}_{56}, \bar{\psi}^{1,\ldots,5}, \bar{\eta}^2\}$$

$$b_3 = \{\psi^\mu, \chi_{56}, w_{12}, w_{34} \mid \bar{w}_{12}, \bar{w}_{34}, \bar{\psi}^{1,\ldots,5}, \bar{\eta}^3\}$$

The boundary conditions for each fermion as defined in each basis vector can be seen by the following table

INSERT TABLE

We have already checked the ABK rules for the basis vectors $\{\mathbb{1}, S, b_1\}$, but now we must check them for b_2 and b_3.

5.2.1 ABK Rules of the NAHE Set

$$b_1 \cdot \mathbb{1} = 4 - 8 = -4 = 0 \mod 4 \checkmark \tag{5.27a}$$

$$b_2 \cdot b_1 = 1 - 5 = -4 = 0 \mod 4 \checkmark \tag{5.27b}$$

$$N_{b_2} b_2 \cdot b_2 = 2 \cdot (4 - 8) = -8 = 0 \mod 8 \checkmark \tag{5.27c}$$

$$N_{b_2} b_2 \cdot S = 2 \cdot 2 = 4 = 0 \mod 4 \checkmark \tag{5.27d}$$

$$b_3 \cdot \mathbb{1} = 4 - 8 = -4 = 0 \mod 4 \checkmark \tag{5.27e}$$

$$b_3 \cdot S = 2 \checkmark \tag{5.27f}$$

$$b_3 \cdot b_1 = 1 - 5 = -4 = 0 \mod 4 \checkmark \tag{5.27g}$$

$$b_3 \cdot b_2 = 1 - 5 = -4 = 0 \mod 4 \checkmark \tag{5.27h}$$

$$b_3 \cdot b_3 = -8 \checkmark \tag{5.27i}$$

We now need to work out the GSO coeffiecients.

5.2.2 GSO Coefficients of the NAHE Set

$$
\begin{array}{c|ccccc}
 & \mathbb{1} & S & b_1 & b_2 & b_3 \\
\hline
\mathbb{1} & -1 & +1 & -1 & \pm1 & \pm1 \\
S & +1 & +1 & +1 & \pm1 & \pm1 \\
b_1 & -1 & +1 & -1 & \pm1 & \pm1 \\
b_2 & \pm1 & \mp1 & \pm1 & \pm1 & \pm1 \\
b_3 & \pm1 & \mp1 & \pm1 & \pm1 & \pm1
\end{array}
$$

There are six new phases that have been introduced, which are the three in the coloums b_2, b_3 and the rows $\mathbb{1}, S, b_1$.

The next step is to analyze the massless spectrum.

5.2.3 Analyzing the Massless Spectrum

First we will consider the NS sector.

- The vector bosons which originated from the states $\psi^\mu \bar{\phi}^a \bar{\phi}^b |0\rangle_{NS}$ in equation (4.6) is now broken by GSO projections to the states

$$
\begin{array}{cccccc}
\psi^\mu_{1/2} & \{\bar{\psi}_{1,\dots,5}\bar{\psi}_{1,\dots,5}\} & \{\bar{\eta}_1\bar{y}_{3,\dots,6}\} & \{\bar{\eta}_2\bar{y}_{1,2}\bar{w}_{5,6}\} & \{\bar{\eta}_3\bar{w}_{1,\dots,4}\} & \{\bar{\phi}_{1,\dots,8}\} \\
 & SO(10) & SO(6)_1 & SO(6)_2 & SO(6)_3 & SO(16)
\end{array} \tag{5.28}
$$

- The fermions

$$(\chi_i, y_i, w_i)\, \partial \bar{X}^\mu |0\rangle_{NS} \tag{5.29}$$

  have all been projected out.

- The scalars defined originally in equation (4.8) after the b_1 GSO projections gave the states

$$\chi_{1,2}\{\bar{w}_{1,\dots,6}, \bar{y}_{1,2}, \bar{\eta}_2, \bar{\eta}_3, \bar{\phi}_{1,\dots,8}\}\{\bar{w}_{1,\dots,6}, \bar{y}_{1,2}, \bar{\eta}_2, \bar{\eta}_3, \bar{\phi}_{1,\dots,8}\} \tag{5.30a}$$

$$\chi_{1,2}\{\bar{\psi}^{1,\dots,5}, \bar{\eta}_1, \bar{y}_{3,4}, \bar{y}_{5,6}\}\{\bar{\psi}^{1,\dots,5}, \bar{\eta}_1, \bar{y}_{3,4}, \bar{y}_{5,6}\} \tag{5.30b}$$

$$\chi_{34,56}\{\bar{\psi}^{1,\dots,5}, \bar{\eta}_1, \bar{y}_{3,4}, \bar{y}_{5,6}\}\{\bar{w}_{1,\dots,6}, \bar{y}_{1,2}, \bar{\eta}_2, \bar{\eta}_3, \bar{\phi}_{1,\dots,8}\} \tag{5.30c}$$

and after the b_2 and b_3 GSO projections the remaining states are

$$
\begin{array}{cc}
\chi^{1,2}\{\bar{\eta}_2, \bar{w}_{5,6}, \bar{y}_{1,2}\}\{\bar{\eta}_3, \bar{w}_{1,\dots,4}, \bar{\phi}_{1,\dots,8}\} & \chi^{1,2}\{\psi_{1,\dots,5}\}\{\bar{\eta}_1, \bar{y}_{3,\dots,6}\} \\
\chi^{3,4}\{\psi_{1,\dots,5}\}\{w_{5,6}, y_{1,2}, \bar{\eta}_2\} & \chi^{3,4}\{\bar{\eta}_1, \bar{y}_{3,\dots,6}\}\{w_{1,\dots,4}, \bar{\eta}_3, \bar{\phi}_{1,\dots,8}\} \\
\chi^{5,6}\{\psi_{1,\dots,5}\}\{w_{1,\dots,4}, \bar{\eta}_3\} & \chi^{5,6}\{\bar{\eta}_1, y_{3,\dots,6}\}\{w_{5,6}, y_{1,2}, \bar{\eta}_2\}
\end{array} \tag{5.31}
$$

which is the massless spectrum from the NS sector.

S Sector

We now want to know how many gravitinos are left from the S sector.

$$
\begin{array}{cccccc}
S: & \{\psi^\mu_{1/2} & \chi_{1,2} & \chi_{3,4} & \chi_{5,6}\} & \partial\bar{X}^\mu \\
 & |\pm\rangle & |\pm\rangle & |\pm\rangle & |\pm\rangle &
\end{array} \tag{5.32}
$$

In the $\mathbb{1}$ sector we find

$$
\begin{array}{cccccc}
 & \{\psi^\mu_{1/2} & \chi_{1,2} & \chi_{3,4} & \chi_{5,6}\} & \partial\bar{X}^\mu \\
\mathbb{1}: & 1 & 1 & 1 & 1 &
\end{array} \tag{5.33}
$$

$$S : \left[\binom{4}{1} + \binom{4}{3}\right]$$

The S sector is the same.
For the b_1 basis vector

$$\begin{array}{lccccc} & \{\psi^{\mu}_{1/2} & \chi_{1,2} & \chi_{3,4} & \chi_{5,6}\} & \partial\bar{X}^{\mu} \\ b_1 : & 1 & 1 & 0 & 0 & \end{array} \tag{5.34}$$

$$\binom{2}{1}\left[\binom{2}{0} + \binom{2}{2}\right]$$

For the b_2 basis vector

$$\begin{array}{lccccc} & \{\psi^{\mu}_{1/2} & \chi_{1,2} & \chi_{3,4} & \chi_{5,6}\} & \partial\bar{X}^{\mu} \\ b_2 : & 1 & 0 & 1 & 0 & \end{array} \tag{5.35}$$

$$\binom{1}{1}\binom{1}{0}\binom{2}{0} + \binom{1}{0}\binom{1}{1}\binom{2}{2}$$

so there is only one gravitino left.
For the b_3 basis vector

$$\begin{array}{lccccc} & \{\psi^{\mu}_{1/2} & \chi_{1,2} & \chi_{3,4} & \chi_{5,6}\} & \partial\bar{X}^{\mu} \\ b_3 : & 1 & 0 & 0 & 1 & \end{array} \tag{5.36}$$

$$\begin{array}{ccc} \binom{1}{1}\binom{1}{0}\binom{2}{0} + \binom{1}{0}\binom{1}{1}\binom{2}{2} & \Rightarrow & C\binom{S}{b_3} = +1 \\ \text{Projected out} & \Rightarrow & C\binom{S}{b_3} = -1 \end{array}$$

Depending on the phase of $C\binom{S}{b_2}$ and $C\binom{S}{b_3}$ we can either project or keep the remaining gravitino.
We can construct tachyon free $\mathcal{N} = 0$ models but the cosmological constant is not equal to zero.

<u>b_1 Sector</u>
The b_1 basis vector has the fermions

$$b_1 = \{\psi^{\mu}_{1,2},\ \chi_{12},\ y_3y_4,\ y_5y_6 \mid \overline{y_3y_4},\ \overline{y_5y_6},\ \bar{\psi}^{\mu}_{1,\dots,5},\ \bar{\eta}_1\} \tag{5.37}$$

and we have the states

$$\begin{array}{lcccccccc} b_1 & \{\psi^{\mu}_{1,2}, & \chi_{12},\ y_3y_4, & y_5y_6 \mid & \overline{y_3y_4}, & \overline{y_5y_6}, & \bar{\psi}^{\mu}_{1,\dots,5}, & \bar{\eta}_1\} & \text{GSO} \\ \mathbb{1} : & \begin{bmatrix} 12 \\ \text{even} \end{bmatrix} & & & & & & & +1 \\ S : & \left[\binom{2}{0} + \binom{2}{2}\right] & \begin{bmatrix} 10 \\ \text{even} \end{bmatrix} & & & & & & +1 \\ b_2 : & 1 & 0 & 0 & 0 & 0 & 1 & 0 & +1 \end{array} \tag{5.38}$$

$$\begin{aligned} &\binom{2}{0}\left[\binom{4}{0} + \binom{4}{2} + \binom{4}{4}\right]\left[\binom{5}{0} + \binom{5}{2} + \binom{5}{4}\right]\binom{1}{0} \\ &+ \binom{2}{0}\left[\binom{4}{1} + \binom{4}{3}\right]\left[\binom{4}{1} + \binom{4}{3}\right]\left[\binom{5}{0} + \binom{5}{2} + \binom{5}{4}\right] \\ &+ \text{other components due to CPT invariance} \end{aligned} \tag{5.39}$$

The terms $\left[\binom{5}{0}+\binom{5}{2}+\binom{5}{4}\right]$ can be interpreted as the 16 generations of $SO(10)$ and the terms $\left[\binom{5}{1}+\binom{5}{3}+\binom{5}{5}\right]$ are the $\bar{1}6$ generations of $SO(10)$.
However, the spectrum is chiral so we only have the 16 generations of $SO(10)$ present.

For the basis vector b_3 we have the exact same result as for b_2 given above in equation (5.38). Once again, we can either keep all the generations from b_1 or we can project them all depending on the relative phase of $C\binom{b_1}{b_2}$ and $C\binom{b_1}{b_3}$.

If

$$C\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = C\begin{pmatrix} b_1 \\ b_3 \end{pmatrix} \tag{5.40}$$

then we have the 16 generations from b_1.

b_2 and b_3 Basis Vectors

For b_2 which has the fermions

$$\{\psi^\mu, \chi_{3,4}, y_{1,2}, w_{5,6} \mid \bar{y}_{1,2}, \bar{w}_{5,6}, \bar{\psi}_{1,\ldots,5}, \bar{\eta}_2\} \tag{5.41}$$

and b_3

$$\{\psi^\mu, \chi_{5,6}, w_{1,2}, w_{3,4} \mid \bar{w}_{1,2}, \bar{w}_{3,4}, \bar{\psi}_{1,\ldots,5}, \bar{\eta}_2\} \tag{5.42}$$

we get a similar result as for the b_1 vector, meaning we get the 16 generations from both b_2 and b_3.

5.2.4 Permutation Symmetry of the b_1, b_2, b_3 Basis Vectors

We should note the permutation symmetry

$$\begin{gathered} b_1 \to b_2 \to b_3 \to b_1 \\ \frac{\{\bar{\eta}_1, \bar{y}_{3,\ldots,6}\}}{\{\chi_{1,2}, y_{3,\ldots,6}\}} \to \frac{\{\bar{\eta}_2, \bar{y}_{1,2}, \bar{w}_{5,6}\}}{\{\chi_{3.4}, y_{1,2}, w_{5,6}\}} \to \frac{\{\bar{\eta}_3, \bar{w}_{1,\ldots,4}\}}{\{\chi_{5,6}, w_{1,\ldots,4}\}} \end{gathered} \tag{5.43}$$

which play an important role.

5.3 Defining ξ Basis Vector

We have 48 generations from the three basis vectors b_1, b_2, b_3. We also have the sectors

$$S + b_1 \qquad ; \qquad S + b_2 \qquad ; \qquad S + b_3 \tag{5.44}$$

which are scalars.

We also have

$$\mathbb{1} + b_1 + b_2 + b_3 \qquad \to \{\bar{\phi}_{1,\ldots,8}\} \text{ periodic} \tag{5.45}$$

and all the others are anti-periodic.

We can define $\xi = \mathbb{1} + b_1 + b_2 + b_3$. By doing this we have the following spin structures

$$C\begin{pmatrix} \xi \\ \mathbb{1} \end{pmatrix} C\begin{pmatrix} \xi \\ S \end{pmatrix} C\begin{pmatrix} \xi \\ b_1 \end{pmatrix} C\begin{pmatrix} \xi \\ b_2 \end{pmatrix} C\begin{pmatrix} \xi \\ b_3 \end{pmatrix} \tag{5.46}$$

The first spin structure can be calculated by

$$C\begin{pmatrix} \xi \\ \mathbb{1} \end{pmatrix} = C\begin{pmatrix} \mathbb{1} + b_1 + b_2 + b_3 \\ \mathbb{1} \end{pmatrix} = e^{\frac{i\pi}{2}\mathbb{1}\cdot\xi} C\begin{pmatrix} \mathbb{1} \\ \xi \end{pmatrix}^* \tag{5.47}$$

As $\mathbb{1} \cdot \xi = 0 - 8 = -8$ we find

$$C\binom{\mathbb{1}}{\xi} = \delta_{\mathbb{1}} C\binom{\mathbb{1}}{\mathbb{1}} C\binom{\mathbb{1}}{b_1+b_2+b_3} = \delta_{\mathbb{1}}^2 C\binom{\mathbb{1}}{\mathbb{1}} C\binom{\mathbb{1}}{b_1} C\binom{\mathbb{1}}{b_2} C\binom{\mathbb{1}}{b_3} \tag{5.48}$$

and therefore

$$\begin{array}{ccccccc} C\binom{\mathbb{1}}{\xi} = & \delta_{\mathbb{1}}^3 & C\binom{\mathbb{1}}{\mathbb{1}} & C\binom{\mathbb{1}}{b_1} & C\binom{\mathbb{1}}{b_2} & C\binom{\mathbb{1}}{b_3} & \\ & (-) & - & - & - & - & = -1 \end{array} \tag{5.49}$$

We will now derive a general formula for the spin structures that are composed of sectors with more than one basis vector. In general we have

$$C\binom{\vec{\alpha}}{\vec{\beta}} = C\binom{m_1a_1 + \ldots + m_na_n}{n_1b_1 + \ldots + n_nb_n}$$

and so

$$\begin{aligned} C\binom{\vec{\alpha}}{n_1b_1 + \ldots + n_nb_n} &= \delta_\alpha C\binom{\vec{\alpha}}{n_1b_1} C\binom{\vec{\alpha}}{n_2b_2 + \ldots + n_nb_n} \\ &= \delta_\alpha C\binom{\vec{\alpha}}{\underbrace{b_1 + \ldots + b_1}} C\binom{\vec{\alpha}}{n_2b_2 + \ldots + n_nb_n} \end{aligned}$$

where the underbraced term is equal to n_1.
Now we consider $C\binom{\vec{\alpha}}{n_1b_1}$ by

$$\begin{aligned} C\binom{\vec{\alpha}}{n_1b_1} &= \delta_\alpha C\binom{\vec{\alpha}}{b_1} C\binom{\vec{\alpha}}{(n_1-1)b_1} \\ &= \delta_\alpha^2 C\binom{\vec{\alpha}}{b_1}^2 C\binom{\vec{\alpha}}{(n_1-2)b_1} \\ &= \ldots \\ &= \delta_\alpha^{(n_1-1)} C\binom{\vec{\alpha}}{b_1}^{(n_1-1)} C\binom{\vec{\alpha}}{(n_1-(n_1-1))b_1} \\ &= \delta_\alpha^{(n_1-1)} C\binom{\vec{\alpha}}{b_1}^{n_1} \end{aligned}$$

and by using this result we therefore find (where the underbraced terms are equal to k)

$$\begin{aligned} C\binom{\vec{\alpha}}{\underbrace{n_1b_1 + \ldots + n_nb_n}} &= \delta_\alpha^{(k-1)} \underbrace{\delta_\alpha^{(n_1-1)} \delta_\alpha^{(n_n-1)}} C\binom{\vec{\alpha}}{b_1}^{n_1} C\binom{\vec{\alpha}}{b_2}^{n_2} \ldots C\binom{\vec{\alpha}}{b_n}^{n_n} \\ &= \delta_\alpha^{(k-k-1+n_1+\ldots+n_n)} C\binom{\vec{\alpha}}{b_1}^{n_1} \ldots C\binom{\vec{\alpha}}{b_n}^{n_n} \\ &= \delta_\alpha^{(-1+n_1+\ldots+n_n)} C\binom{\vec{\alpha}}{b_1}^{n_1} \ldots C\binom{\vec{\alpha}}{b_n}^{n_n} \end{aligned}$$

Using rule (3) for the one loop phases, we find

$$\begin{aligned} C\begin{pmatrix}\vec{\alpha}\\ b_j\end{pmatrix} &= e^{\frac{i\pi}{2}\alpha\cdot b_j} C\begin{pmatrix}b_j\\ \vec{\alpha}\end{pmatrix}^{*} \\ &= e^{\frac{i\pi}{2}\alpha\cdot b_j}\ \delta_{b_j}^{(-1+m_1+\ldots+m_n)} \prod_{i=1}^{n} C\begin{pmatrix}b_j\\ b_i\end{pmatrix}^{*m_i} \end{aligned}$$

This leads to our final result, which is

$$C\begin{pmatrix}\vec{\alpha}\\ \vec{\beta}\end{pmatrix} = \delta_{\alpha}^{(-1+n_1+\ldots+n_n)} \prod_{j=1}^{n} e^{\frac{i\pi}{2}\alpha\cdot b_j\cdot n_j}\ \delta_{b_j}^{((-1+m_1+\ldots+m_n)n_j)} \prod_{i=1}^{n} C\begin{pmatrix}b_j\\ b_i\end{pmatrix}^{*m_i} \tag{5.50}$$

Again, this result is simple to put into a computer and therefore calculating the results can be done swiftly when utilizing this method.

5.3.1 GSO Projections of the ξ Basis Vector

We can now perform the GSO projections of the ξ basis vector with the other basis vectors. We find

$$C\begin{pmatrix}\xi\\ \mathbb{1}\end{pmatrix} = -1 \tag{5.51a}$$

$$\begin{aligned} C\begin{pmatrix}\xi\\ S\end{pmatrix} &= e^{\frac{i\pi}{2}\xi\cdot S} C\begin{pmatrix}S\\ \mathbb{1}+b_1+b_2+b_3\end{pmatrix} \\ &= \underset{-1}{\delta_S^3}\quad \underset{+1}{C\begin{pmatrix}S\\ 1\end{pmatrix}}\quad \underset{+1}{C\begin{pmatrix}S\\ b_1\end{pmatrix}}\quad \underset{+1}{C\begin{pmatrix}S\\ b_2\end{pmatrix}}\quad \underset{+1}{C\begin{pmatrix}S\\ b_3\end{pmatrix}}\quad = -1 \end{aligned} \tag{5.51b}$$

$$\begin{aligned} C\begin{pmatrix}\xi\\ b_1\end{pmatrix} &= e^{\frac{i\pi}{2}\xi\cdot b_1} C\begin{pmatrix}b_1\\ \mathbb{1}+b_1+b_2+b_3\end{pmatrix} \\ &= \underset{-1}{\delta_{b_1}^3}\quad \underset{-1}{C\begin{pmatrix}b_1\\ \mathbb{1}\end{pmatrix}}\quad \underset{-1}{C\begin{pmatrix}b_1\\ b_1\end{pmatrix}}\quad \underset{-1}{C\begin{pmatrix}b_1\\ b_2\end{pmatrix}}\quad \underset{-1}{C\begin{pmatrix}b_1\\ b_3\end{pmatrix}}\quad = -1 \end{aligned} \tag{5.51c}$$

Similarly,

$$C\begin{pmatrix}\xi\\ b_2\end{pmatrix} = -1 \qquad \text{and} \qquad C\begin{pmatrix}\xi\\ b_3\end{pmatrix} = -1 \tag{5.51d}$$

Now we can perform the GSO projection of the states which adhearing to the needed results from equations (5.51a,b,c,d), for the vectors:

$$\begin{array}{cccccc} \xi & & \bar{\phi}_1 & \ldots & \bar{\phi}_8 \\ & \psi^\mu & \lvert\pm\rangle & \ldots & \lvert\pm\rangle \\ \vec{1} & \vec{1} & 1 & \ldots & 1 \\ & \psi^\mu : & \left[\begin{pmatrix}8\\0\end{pmatrix} + \right. & \begin{pmatrix}8\\2\end{pmatrix} + \ldots & \left.\begin{pmatrix}8\\8\end{pmatrix}\right] \\ S & 1 & & \vec{0} & \\ b_{1,2,3} & 1 & & \vec{0} & \end{array} \tag{5.52}$$

These states transform as $2^7 = 128$ of the $SO(16)$ group.

$$120 + 128 = 248 \quad \rightarrow \quad SO(16) \quad \rightarrow \quad E_8 \tag{5.53}$$

For the scalars:

$$\begin{array}{cccccc} & \{\chi, y, w\} & |\pm\rangle & & \dots & |\pm\rangle \\ \vec{1}: & \{\} & \Big[\binom{8}{0} + & \binom{8}{2} & + \dots & \binom{8}{8}\Big] \\ & & & & & \\ S & \chi_{1,\dots,6} & [& & &] \\ b_1 & \chi_{1,2} & [& & &] \\ b_{2,3} & \text{are all} & \text{projected out} & & & \end{array} \tag{5.54}$$

5.4 Extension of the NAHE Set

Before moving on to the case of the three generation models, we will first mention a simple extension of the NAHE. This is not directly relevent to the realistic models but it does illustrate the structure of these models and the relation between $(2,2)$ and $(2,0)$ model. This extension is obtained by adding the basis vector

$$x = \{\bar{\psi}_{1,\dots,5}, \bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_3\} = 1 \tag{5.55}$$

and all the other fermions are anti-periodic. We can fix the phases appropriately but we will not cover how here.

With this additional basis vector and the appropriate choice of phases, the gauge group is

$$E_6 \times U(1)^2 \times SO(4)^3 \times E_8 \tag{5.56}$$

The basis vector X produced the $16 + \bar{1}6$ in $78 = 45 + 16 + \bar{1}6 + 1$ and enhances the gauge group to E_6.

The $SO(4)^3$ groups are produced by

$$\begin{array}{ccc} \{\bar{y}_{3,\dots,6}\} & \{\bar{y}_{1,2}\bar{w}_{5,6}\} & \{\bar{w}_{1,\dots,4}\} \\ SO(4)_1 & SO(4)_2 & SO(4)_3 \end{array} \tag{5.57}$$

The three world sheet complex fermions $\bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_3$ produce the three $U(1)$ currents.

One combination is embedded in E_6, *i.e*

$$U(1)_{E_6} = U(1)_{\bar{\eta}_1} + U(1)_{\bar{\eta}_2} + U(1)_{\bar{\eta}_3} \tag{5.58}$$

and we have two orthogonal combinations

$$U(1)_{\bar{\eta}_1} - U(1)_{\bar{\eta}_2} \tag{5.59a}$$

$$U(1)_{\bar{\eta}_1} + U(1)_{\bar{\eta}_2} - 2U(1)_{\bar{\eta}_3} \tag{5.59b}$$

The important thing that we must note is that the x basis vector splits $\{\bar{y}, \bar{w}\}$ from $\{\bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_3\}$.

There is a symmetry between the left and right movers. We have $\mathcal{N} = 2$ in the right and left moving sectors. The two supercurrents are

$$T_{F_L} = e^{i\chi_{i,i+1}}(y_i w_i \pm i y_{i+1} w_{i+1}) \tag{5.60a}$$

$$T_{F_R} = e^{i\bar{\eta}_i}(\bar{y}_i \bar{w}_i + i\bar{y}_{i+1}\bar{w}_{i+1}) \tag{5.60b}$$

and the $U(1)$ currents are generated in $\mathcal{N}=2$ by

$$U(1)_L = \chi_{1,2} + \chi_{3,4} + \chi_{5,6} \tag{5.61a}$$

$$U(1)_R = \bar{\eta}_1 + \bar{\eta}_2 + \bar{\eta}_3 \tag{5.61b}$$

and for any choice of boundary conditions, the $\mathcal{N}=2$ supercurrent T_F is well defined.

However, we can generate this model in a different way.
We can start with

$$\{\mathbb{1}, S, X, \xi = \mathbb{1} + b_1 + b_2 + b_3\} \tag{5.62}$$

and make an appropriate choice of phases to give

$$\mathcal{N}=4 \qquad\qquad SO(12)\times E_8\times E_8 \tag{5.63}$$

We can now twist this by b_1 and b_2 to obtain

$$\mathcal{N}=1 \qquad\qquad SO(4)^3\times E_6\times U(1)^2\times E_8 \tag{5.64}$$

which is the same group. This is infact the same way that we can choose any set of basis vectors as long as they produce the same additive group and as long as we are careful with the phases.

5.5 Matter Sector

To be complete, we should mention what happens in the matter sector.

For example, we can examine b_1 after the projection of the NAHE set. The vacuum was

$$\begin{aligned}
&\binom{2}{0}\left[\binom{4}{0}+\binom{4}{2}+\binom{4}{4}\right]\left[\binom{5}{0}+\binom{5}{2}+\binom{5}{4}\right]\binom{1}{0}\\
&+\binom{2}{0}\left[\binom{4}{1}+\binom{4}{3}\right]\left[\binom{4}{1}+\binom{4}{3}\right]\left[\binom{5}{0}+\binom{5}{2}+\binom{5}{4}\right]\\
&+\text{ other components due to CPT invariance}
\end{aligned} \tag{5.65}$$

Since $x=\{\vec{0}_L \mid \vec{0}_R\,\underbrace{\bar{\psi}_1^{1,\ldots,5},\bar{\eta}_1,\bar{\eta}_2,\bar{\eta}_3},\vec{0}_R\}$ (where the underbraced term have 1) then we can see in the b_1 vacuum that we have either an even or odd number of $|-\rangle$ among the fermions which are periodic in x. Therefore for any choice of GSO projection half of these are projected out.

We will not go into any more detail here but the spectrum can be worked out from this point.

5.6 Continuing the Extension of the NAHE Set

We find that

$$b_j + X \rightarrow 8\times(10_j + 1_j + 1_j) + (E_8\times E_8) \tag{5.66}$$

The $10+1$ combine with the 16 of $SO(10)$ to form the 27 of E_6.
So we have

$$8\times 3(27) \text{ of } E_6 \quad = \quad 24 \text{ generations} \tag{5.67}$$

The other 1_j are singlets of E_6 and are the twisted moduli with an equal number to the number of generations. Also we find that the scalar representations from the NS sector are completed to $3(27+\bar{27})$ of E_6 and equal the number of untwisted moduli.
There are also $E_8 \times E_8$ singlets charged under $SO(4)^3$.

The final remark that should be made is that we can project the E_6 generators by the choice of GSO projection

$$C\begin{pmatrix}\xi\\ x\end{pmatrix} = \pm 1 \tag{5.68}$$

One of these choices will produce the model that was just described.
The other choice will project the generators in ξ and x that produce the 128 in $E_8 \times E_8$. This can be seen by considering the GSO projection

$$e^{i\pi\xi\cdot F_X} = \delta_x \; C\begin{pmatrix}x\\ \xi\end{pmatrix}|s\rangle_x \tag{5.69}$$

as the exponential gives a $+1$ contribution and the phases on the right hand side give a -1 contribution, therefore projecting all of the states out.

The gauge group that we then obtain is

$$\mathcal{N}=4 \qquad SO(12)\times SO(16)\times SO(16)$$

When we apply the b_1, b_2 projections we get

$$\mathcal{N}=4 \qquad SO(4)^3\times SO(10)\times U(1)^3\times SO(16)$$

with 24×16 of $SO(10)$ and $3(10+10)+(1,1)$.

A similar thing operates in the fermionic three generation models. The important thing is that the x projection splits the $\{\bar{y},\bar{w}\}$ (compactified space) from the $\{\bar{\eta}\}$.

The degeneracy of the Ramond vacua under the $\{y,w|\bar{y},\bar{w}\}$ produces the multiplicity of the generations.

The models 'remember' some of their $(2,2)$ structure. The important thing for the extensions is the symmetry between $\{y,w|\bar{y},\bar{w}\}$. The structure that has just been elaborated on is the key to the Gepner construction. The Gepner construction could also split the gauge degrees of freedom from the internal CFT degrees of freedom and find modular invariant solutions for the more complicated minimal models. Now we see that also in the $(2,0)$ this structure can be preserved.

Chapter 6

Generation Models

We have analyzed the full massless spectrum of the NAHE set.
We have:

- $\mathcal{N} = 1$ SUSY
- $SO(10)$ observable gauge group
- 48 generations

The construction of the realistic models is really only starting after the NAHE set.

The $\bar{\psi}^{1,\ldots,5}$ have the gauge group $SO(10)$. The states are

$$\psi^\mu \bar{\psi}^{1,\ldots,5} \bar{\psi}^{1,\ldots,5} \left|0\right\rangle_{NS} \tag{6.1}$$

which can be split to

$$\begin{array}{lll} SO(6): & \psi^\mu \bar{\psi}^{1,\ldots,3} \bar{\psi}^{1,\ldots,3} & \left|0\right\rangle_{NS} \\ SO(4): & \psi^\mu \bar{\psi}^{4,5} \bar{\psi}^{4,5} & \left|0\right\rangle_{NS} \end{array} \tag{6.2}$$

and therefore the gauge group $SO(10)$ breaks to $SO(6) \times SO(4)$.
By considering the GSO projection

$$e^{i\pi\alpha\cdot F} \left|s\right\rangle_{NS} = \delta_\alpha \tag{6.3}$$

and supposing $\delta_\alpha = -1 \Leftrightarrow \alpha(\phi^\mu) = 1$ then the states

$$\begin{array}{l} \psi^\mu \bar{\psi}^{1,\ldots,5} * \ \bar{\psi}^{1,\ldots,5} * \\ \psi^\mu \ \bar{\psi}^{1,\ldots,5} * \ \bar{\psi}^{1,\ldots,5} \\ \psi^\mu \ \bar{\psi}^{1,\ldots,5} \ \bar{\psi}^{1,\ldots,5} \end{array} \tag{6.4}$$

(where the $*$ denotes the complex conjugate) have the following GSO projections:

- $\psi^\mu \bar{\psi}^{1,\ldots,5} * \ \bar{\psi}^{1,\ldots,5} *$

$$e^{i\pi\left(1-\{\frac{1}{2}(-1)+\frac{1}{2}(-1)\}\right)} = e^{i\pi(1+1)} = +1x \qquad x \neq \delta_\alpha \tag{6.5}$$

- $\psi^\mu\ \bar{\psi}^{1,\ldots,5}\ \bar{\psi}^{1,\ldots,5}$

$$e^{i\pi\left(1-\{\frac{1}{2}(1)+\frac{1}{2}(1)\}\right)} = e^{i\pi(1-1)} = +1x \qquad x \neq \delta_\alpha \tag{6.6}$$

- $\psi^\mu\ \bar{\psi}^{1,\ldots,5} * \ \bar{\psi}^{1,\ldots,5}$

$$e^{i\pi\left(1-\{\frac{1}{2}(-1)+\frac{1}{2}(+1)\}\right)} = e^{i\pi} = -1x \qquad x = \delta_\alpha \checkmark \tag{6.7}$$

So only the state $\psi^\mu\ \bar{\psi}^{1,\ldots,5} * \ \bar{\psi}^{1,\ldots,5}$ survives.
This can be interpreted as the $SO(10)$ gauge group breaking to $U(5) \equiv SU(5) \times U(1)$.

Suppose we have

$$\begin{array}{c} \\ \alpha \\ \beta \end{array}\begin{array}{c} \begin{array}{ccccc} \bar{\psi}_1 & \bar{\psi}_2 & \bar{\psi}_3 & \bar{\psi}_4 & \bar{\psi}_5 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{array} \tag{6.8}$$

then

$$\alpha \Rightarrow SO(10) \rightarrow SO(6) \times SO(4) \tag{6.9a}$$

$$\beta \Rightarrow SO(6) \times SO(4) \rightarrow SU(3) \times U(1)_{B-L} \times SU(2) \times U(1)_{T_3} \tag{6.9b}$$

This is how the gauge groups are obtained in the FFF.
Therefore, we can construct models with

$$\begin{aligned} &SU(5) \times U(1) \\ &SO(6) \times SO(4) \\ &SU(3) \times SU(2) \times U(1)^2 \end{aligned} \tag{6.10}$$

gauge groups.

We now need to consider the number of generations there are.
We will only analyze one of the sectors of b_1 and b_2 to see how the generations can be obtained.
The number of generations is controlled by the boundary conditions of $\{y, w|\bar{y}, \bar{w}\}$

1. The boundary conditions of $\{\psi^\mu, \chi_{1,2}, \chi_{3,4}, \chi_{5,6}\}$ are fixed by requiring $\mathcal{N} = 2$ SUSY

2. We do not want to break $SU(3) \times SU(2)$, this means we want

$$\bar{\psi}^{1,\ldots,5} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ & \text{or} & / & \text{and} & \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

3. The number of generations is reduced by adding three additional basis vectors

We should remark that these boundary condition allow 12 right movers which can be real of complexified to form 6 complex fermions. So for these choices there is no significance. These pairings play an important role in the phenomenology of models.

So we can choose the following pairing

$$\{y^3\bar{y}^3, y^4\bar{y}^4, y^5\bar{y}^5, y^6\bar{y}^6 \mid y^1\bar{y}^1, y^2\bar{y}^2, w^5\bar{w}^5, w^6\bar{w}^6 \mid w^1\bar{w}^1, w^2\bar{w}^2, w^3\bar{w}^3, w^4\bar{w}^4\} \tag{6.11}$$

6.1 Three Generation Model

We will now construct a three generation model which has the Standard Model gauge group.
We need three additional basis vectors.

As we saw in the discussion of the E_6 model, half of the generations are projected by the vector x.

In the three generation model we have something similar.
We have a basis vector with

$$\gamma : \quad \to \quad \gamma\{\psi^{1,\dots,5}, \bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_3, \bar{\phi}^{1,\dots,4}\} = \frac{1}{2} \tag{6.12a}$$

$$\gamma\{\text{others}\} = \{0, 1\} \tag{6.12b}$$

Then we have

$$2\gamma\{\psi^{1,\dots,5}, \bar{\eta}_{1,2,3}, \bar{\phi}^{1,\dots,4}\} = 1 \tag{6.13a}$$

$$2\gamma\{\text{others}\} = 0 \tag{6.13b}$$

The effect of the $2\vec{\gamma}$ vector is the project half of the generations in the same way that the vector X did. This will choose the sign under $U(1)_{\bar{\eta}_j}$ to be either $\pm 1/2$.

The remaining degeneracies of the zero modes is due to the degeneracy under $\{y, w | \bar{y}, \bar{w}\}$.
The NAHE splits into three groups

$$\begin{array}{ccccc} & \{y^{3,\dots,6} & | & \bar{y}^{3,\dots,6}\} & \\ b_1 & 1 & 1 & 1 & 1 \\ b_2 & 0 & 0 & 0 & 0 \\ b_3 & 0 & 0 & 0 & 0 \end{array} \tag{6.14a}$$

$$\Rightarrow \left[\binom{4}{0} + \binom{4}{2} + \binom{4}{4}\right]_{b_1}$$

$$\begin{array}{ccccc} & \{y^1y^2, w^5w^6 & | & \bar{y}^1\bar{y}^2, \bar{w}^5\bar{w}^6\} & \\ b_1 & 0 & 0 & 0 & 0 \\ b_2 & 1 & 1 & 1 & 1 \\ b_3 & 0 & 0 & 0 & 0 \end{array} \tag{6.14b}$$

$$\Rightarrow \left[\binom{4}{0} + \binom{4}{2} + \binom{4}{4}\right]_{b_2}$$

$$\begin{array}{ccccc} & \{w^{1\dots,4} & | & \bar{w}^{1,\dots,4}\} & \\ b_1 & 0 & 0 & 0 & 0 \\ b_2 & 0 & 0 & 0 & 0 \\ b_3 & 1 & 1 & 1 & 1 \end{array} \tag{6.14c}$$

$$\Rightarrow \left[\binom{4}{0} + \binom{4}{2} + \binom{4}{4} \right]_{b_3}$$

To reduce the number of generations we need to reduce this degeneracy by using additional boundary condition basis vectors.