UNIVERSITY OF LIVERPOOL

MATH 423 January 2011

EXAMINER: Prof. A.E. Faraggi, EXTENSION 43774.

TIME ALLOWED: Two and a half hours

Full marks can be obtained for complete answers to FIVE questions. Only the best FIVE answers will be counted.

You may use a university approved pocket calculator and the constants:

$$c = 3 \times 10^{10}\frac{\text{cm}}{s}; \qquad \hbar = 1.054 \times 10^{-27} erg \cdot s;$$

$$G_N = 6.674 \times 10^{-8}\frac{cm^3}{g \cdot s^2} \qquad m_e = 9.109 \times 10^{-28} g;$$

$$m_p = 1.672 \times 10^{-24} g; \qquad k = 1.380 \times 10^{-16}\frac{erg}{K};$$

$$m_{\text{Planck}} = 2.17 \times 10^{-5} g; \qquad m_{\text{sun}} \approx 2 \times 10^{33} g;$$

1. Consider the infinitesimal line element,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - dx^2.$$

(a) Write the metric $g_{\mu\nu}$ and its inverse in an explicit in matrix form.

[2 marks]

(b) Find the set of independent transformations of the form

$$\begin{aligned} t &\to t + \epsilon A(t, x) \\ x &\to x + \epsilon B(t, x) \ , \end{aligned}$$

where ϵ is an infinitesimal constant and the functions A and B have to be determined by the requirement that ds^2 is invariant. State what each transformation represents in space time.

[18 marks]

2. The electromagnetic field strength tensor is given by:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}$$

where $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields respectively. We define the tensor $T_{\lambda\mu\nu}$ by

$$T_{\lambda\mu\nu} = \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu}$$

(a) Derive the source free Maxwell equations.

[10 marks]

Additionally the electromagnetic current four vector is given by

$$j^\mu = (c\rho, j^1, j^2, j^3)$$

(b) Derive the Maxwell equations with sources.

[10 marks]

3.

(a) The vacuum energy associated with the current acceleration of the universe is $\rho_{\text{vac}} = 7.7 \times 10^{-27}\text{kg/m}^3$. Derive a fundamental length scale, ℓ_{vac} associated with this vacuum energy in terms of ρ_{vac}, the Planck constant $\hbar$, and the speed of light c. Express the numerical value for ℓ_{vac} in μm where $1\mu\text{m} = 10^{-6}\text{m}$.

[10 marks]

(b) The Standard Bohr radius is $a_0 = \frac{\hbar^2}{me^2} \approx 5.29 \times 10^{-9}\text{cm}$, and arises from the electric potential $V = -\frac{e^2}{r}$. What would be the gravitational Bohr radius if the attraction force binding the electron to the proton was gravitational?

[10 marks]

4. Consider the action

$$S[x,e] = \frac{1}{2}\int e d\tau \left(\frac{1}{e^2} \left(\frac{dx^\mu}{d\tau} \right)^2 - m^2 \right) , \qquad (1)$$

where τ is an arbitrary parameter and $e d\tau$ is an invariant line element.

(a) Perform variations of x^μ and of e to obtain the equations of motion for x^μ and e respectively.

[6 marks]

(b) For $m^2 > 0$, eliminate e by its equation of motion, and substitute the result back into (1). Show that you obtain the action for a free massive particle.

[7 marks]

(c) Instead of eliminating e by its equation of motion, you can set it to a constant value by reparameterisation. Derive the equation of motion and the contraint equation for the two cases $m^2 > 0$ and $m^2 = 0$.

[7 marks]

5. The Polyakov action is given by:

$$S_{\mathrm{P}} = -\frac{T}{2}\int d^2\sigma\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu} \ ,$$

where $h = \det(h_{\alpha\beta})$.

(a) For a 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad \delta A = \begin{pmatrix} \delta a_{11} & \delta a_{12} \\ \delta a_{21} & \delta a_{22} \end{pmatrix}$$

show that

$$\delta \mathrm{detA} = \mathrm{detATr}(A^{-1}\delta A) \ .$$

[4 marks]

(b) Hence, derive from the Polyakov action the world–sheet energy momentum tensor $T_{\alpha\beta}$, which is defined by

$$T_{\alpha\beta} = -\frac{2}{T}\frac{1}{\sqrt{-h}}\frac{\delta S}{\delta h^{\alpha\beta}}$$

[8 marks]

(c) In the conformal gauge $h_{\alpha\beta} = \eta_{\alpha\beta}$, the energy–momentum tensor takes the form

$$T_{\alpha\beta} = \partial_\alpha X^\mu\partial_\beta X_\mu - \frac{1}{2}\eta_{\alpha\beta}\eta^{\gamma\delta}\partial_\gamma X^\mu\partial_\delta X_\mu \ .$$

Show that

$$\begin{aligned} \eta^{\alpha\beta}T_{\alpha\beta} &= 0 \\ \partial^\alpha T_{\alpha\beta} &= 0 \end{aligned}$$

[8 marks]

6. The equations of motion for a relativistic string in the conformal gauge are

$$(\partial_0^2 - \partial_1^2)X^\mu = 0$$

Solutions must satisfy the constraints

$$\partial_0 X^\mu \partial_1 X_\mu = 0\ , \quad \partial_0 X^\mu \partial_0 X_\mu + \partial_1 X^\mu \partial_1 X_\mu = 0$$

We consider closed strings with boundary conditions

$$X^\mu(\sigma^0, \sigma^1) = X^\mu(\sigma^0, \sigma^1 + \pi)$$

(a) Show that equations of motion and constraints are solved by

$$\begin{aligned} X^0 &= 2R\sigma^0 \\ X^1 &= R\cos(2\sigma^1)\cos(2\sigma^0) \\ X^2 &= R\sin(2\sigma^1)\cos(2\sigma^0) \\ X^i &= 0 \text{ for } i > 2 \end{aligned}$$

Describe in words how the closed string moves.

[7 marks]

(b) Compute the length, the mass and the momentum of the string.

[7 marks]

(c) Show that the string is at rest at time $X^0 = 0$. Express the total energy in terms of its length at time $X^0 = 0$ and interpret the result.

[6 marks]

7. Consider the "lightlike" compactification, in which we identify events with position and time coordinates related by

$$\begin{pmatrix} x \\ ct \end{pmatrix} \sim \begin{pmatrix} x \\ ct \end{pmatrix} + 2\pi \begin{pmatrix} R \\ -R \end{pmatrix} \tag{2}$$

(a) Rewrite this identification using light–cone coordinates.

[5 marks]

(b) Consider coordinates (ct', x') related to (ct, x) by a boost with veclocity parameter β. Express the identifications in terms of the primed coordinates.

[7 marks]

(c) Consider the family of indentifications given by

$$\begin{pmatrix} x \\ ct \end{pmatrix} \sim \begin{pmatrix} x \\ ct \end{pmatrix} + 2\pi \begin{pmatrix} \sqrt{R^2 + R_s^2} \\ -R \end{pmatrix} . \tag{3}$$

Show that there is a boosted frame S' in which the identification (3) becomes a standard identification (*i.e.* the space coordinate is identified but the time coordinate is not).

[8 marks]