UNIVERSITY OF LIVERPOOL

MATH 423 January 2010

EXAMINER: Prof. A.E. Faraggi, EXTENSION 43774.

TIME ALLOWED: Two and a half hours

Full marks can be obtained for complete answers to FIVE questions. Only the best FIVE answers will be counted.

You may use a university approved pocket calculator and the constants:

$$c = 3\times 10^{10}\frac{\text{cm}}{s}; \qquad \hbar = 1.054\times 10^{-27} erg\cdot s;$$

$$G_N = 6.674\times 10^{-8}\frac{cm^3}{g\cdot s^2} \qquad m_e = 9.109\times 10^{-28} g;$$

$$m_p = 1.672\times 10^{-24} g; \qquad k = 1.380\times 10^{-16}\frac{erg}{K};$$

$$m_{\text{Planck}} = 2.17\times 10^{-5} g; \qquad m_{\text{sun}} \approx 2\times 10^{33} g;$$

1. The equations of motion for a relativistic string in the conformal gauge are

$$(\partial_0^2 - \partial_1^2)X^\mu = 0 \ .$$

The solutions must satisfy the constraints

$$\partial_0 X^\mu \partial_1 X_\mu = 0 \ , \quad \partial_0 X^\mu \partial_0 X_\mu + \partial_1 X^\mu \partial_1 X_\mu = 0 \ .$$

We consider open strings with boundary conditions

$$\partial_1 X_\mu = 0 \text{ for } \sigma^1 = 0, \pi \ .$$

(a) Show that the equations of motions, the constraints and the boundary conditions are satisfied by

$$\begin{aligned} X^0 &= L\sigma^0 \\ X^1 &= L\cos\sigma^1 \cos\sigma^0 \\ X^2 &= L\cos\sigma^1 \sin\sigma^0 \\ X^i &= 0 \text{ for } i > 2 \end{aligned}$$

Explain in words how the string moves in this solution.

[7 marks]

(b) Compute the mass, momentum and angular momentum of the string.

[7 marks]

(c) Compute the speed of the endpoints of string. Explain why the result you find holds for any solution of the open string theory.

[6 marks]

2. (a) Give the Lorentz transformations for the components a_μ of a vector under a boost along the x^1 axis.

[7 marks]

(b) Show that the object $\frac{\partial}{\partial x^\mu}$ transform under a boost along the x^1 axis as the a_μ vector considered in (a) does.

[7 marks]

(c) Show that, in quantum mechanics, the expression for the energy and momentum in terms of derivatives can be written compactly as $p_\mu = \frac{\hbar}{i}\frac{\partial}{\partial x^\mu}$.

[6 marks]

3. (a) Consider the plane (x, y) with the identification

$$(x, y) \sim (x + 2\pi R, y + 2\pi R) \ .$$

What is the resulting space?

[6 marks]

(b) Consider the circle S^1, presented as the real line with the identifications $x \sim x + 2$. The circle is the space $-1 < x \leq 1$ with the points $x = \pm 1$ identified. The orbifold S^1/Z_2 is defined by imposing the Z_2 identification $x \sim -x$. Show that there are two points on the circle that are left fixed by the Z_2 action.

[7 marks]

(c) Consider a torus T^2, presented as the (x, y) plane with the identifications $x \sim x+2$ and $y \sim y+2$. Choose $-1 < x, y \leq -1$ as the fundamental domain. The orbifold T^2/Z_2 is defined by imposing the Z_2 identification $(x, y) \sim (-x, -y)$. Prove that there are four points on the torus that are left fixed by the Z_2 transformation.

[7 marks]

4. (a) The Standard Bohr radius is $a_0 = \frac{\hbar^2}{me^2} \approx 5.29 \times 10^{-9}$cm, and arises from the electric potential $V = -\frac{e^2}{r}$. What would be the gravitational Bohr radius if the attraction force binding the electron to the proton was gravitational?

[9 marks]

(b) In units where G, c and $\hbar$ are set equal to one, the temperature of a black hole is given by $kT = \frac{1}{8\pi M}$. Insert back the factors of G, c and $\hbar$ into this formula. Evaluate the temperature of a black hole of a million solar masses. What is the mass of a black hole whose temperature is room temperature?

[11 marks]

5. A string with tension T_0 is stretched from $x = 0$ to $x = 2a$. The part of the string $x \in (0, a)$ has constant mass density μ_1 and the part of the string $x \in (a, 2a)$ has constant mass density μ_2. Consider the differential equation

$$\frac{d^2y}{dx^2} + \frac{\mu(x)}{T_0}\omega^2 y(x) = 0.$$

that determines the normal oscillations.

(a) What boundary conditions should be imposed on $y(x)$ and $\frac{dy}{dx}(x)$ at $x = a$?

[5 marks]

(b) Write the conditions that determine the possible frequencies of oscillation.

[15 marks]

6. (a) The action for a relativistic free particle of mass m is given by

$$S = -mc \int_{\mathcal{P}} ds = -mc \int_{\tau_i}^{\tau_f} \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} d\tau \ ,$$

where c is the speed of light. Derive the equation of motion from this action by taking the variation $x^\mu(\tau) \to x^\mu(\tau) + \delta x^\mu(\tau)$.

[10 marks]

(b) The action for a relativistic particle of mass m and charge q coupled to an electromagentic field is given by

$$S = -mc \int_{\mathcal{P}} ds + \frac{q}{c} \int_{\mathcal{P}} d\tau A_\mu(x(\tau)) \frac{dx^\mu}{d\tau}(\tau) \ .$$

Derive the equation of motion from this action by taking the variation $x^\mu(\tau) \to x^\mu(\tau) + \delta x^\mu(\tau)$.

[10 marks]

7. The action for the relativistic string is given by (with $c = 1$)

$$S_{\text{NG}}[X] = \int d^2\sigma \mathcal{L} = -T \int_\Sigma d^2\sigma \sqrt{|det(\partial_\alpha X^\mu \partial_\beta X_\mu)|} \ .$$

(a) Show that the action is invariant under reparametrisations of the world–sheet Σ:

$$\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma^0, \sigma^1), \quad \text{where } \det\left(\frac{\partial \tilde{\sigma}^\alpha}{\partial \sigma^\beta}\right) > 0 \ .$$

[10 marks]

(b) Compute the momentum densities

$$P^0_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} \quad , \quad P^1_\mu = \frac{\partial \mathcal{L}}{\partial X'^\mu} \ .$$

[5 marks]

(c) Show that the canonical momenta $\Pi^\mu = P^\mu_0$ are subject to the two constraints

$$\begin{aligned} \Pi^\mu X'_\mu &= 0 \ , \\ \Pi^2 + T^2 (X')^2 &= 0 \ , \end{aligned}$$

and that the Hamiltonian vanishes

$$\mathcal{H} = \dot{X}\Pi - \mathcal{L} = 0 \ .$$

[5 marks]