MATH425 - Quantum Field Theory

Set Work: Sheet 9

1. The current $j^\mu = \overline{\psi}\gamma^\mu\psi$ is conserved, i.e. satisfies $\partial_\mu j^\mu = 0$. In relativistic quantum mechanics we interpreted $\int j^0 d^3\mathbf{x}$ as the probability. Now we are going to interpret $Q = \int : j^0 : d^3\mathbf{x}$ as the total charge (electric charge if ψ describes electrons). Show that,

$$Q = \int d^3\mathbf{p}[N_a(\mathbf{p}) - N_b(\mathbf{p})],$$

where

$$N_a(\mathbf{p}) = \frac{1}{(2\pi)^3}\frac{1}{2p^0}a_r^\dagger(\mathbf{p})a_r(\mathbf{p}),$$

and similarly for $N_b(\mathbf{p})$. We interpret $N_a(\mathbf{p})$, $N_b(\mathbf{p})$ as the number density of particles and antiparticles respectively; so this indicates that particles and antiparticles have equal and opposite charges.

2. Consider the Lagrangian density

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I,$$

with

$$\mathcal{L}_0 = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2$$
$$\mathcal{L}_I = -\frac{\lambda_3}{3!}\phi^3 - \frac{\lambda_4}{4!}\phi^4 \ .$$

(i) Derive the equation of motion from

$$\partial^\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\right] - \frac{\partial\mathcal{L}}{\partial\phi} = 0.$$

(ii) With the generalized momentum $\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}}$, write down the Hamiltonian defined by

$$H = \int[\pi\dot{\phi} - \mathcal{L}]d^3\mathbf{x} \ .$$

(iii) Show that if we take the usual canonical commutation relations,

$$[\phi(\mathbf{x},t),\pi(\mathbf{x}',t)] = i\hbar\delta(\mathbf{x}-\mathbf{x}'),$$
$$[\phi(\mathbf{x},t),\phi(\mathbf{x}',t)] = 0,$$
$$[\pi(\mathbf{x},t),\pi(\mathbf{x}',t)] = 0,$$

the above equation of motion is also obtained from

$$i\dot{\pi} = [\pi, H] \ .$$