MATH425 - Quantum Field Theory
Set Work: Sheet 6

1.

Consider a plane wave

$$\psi(x) = v(p)e^{ip.x}$$

where

$$p^0 = \sqrt{\mathbf{p}^2 + m^2}.$$

Show that this will be a solution to the Dirac equation if

$$v(p) = N(p)\begin{pmatrix} \frac{\sigma.\mathbf{p}}{p^0+m}\chi \\ \chi \end{pmatrix},$$

where χ is a 2-vector and $N(p)$ is arbitrary. Suppose we pick two orthonormal 2-vectors $\chi_{1,2}$ satisfying

$$\chi_r^\dagger \chi_s = \delta_{rs}.$$

Show that if we pick $N(p) = \sqrt{p^0 + m}$, then we have

$$\overline{v_r(p)}v_s(p) = -2m\delta_{rs}.$$

2. We know that positive energy solutions of the Dirac equation are given by $\psi_p^{(r)}(x) = e^{-ip.x}u_r(p)$, where

$$u_r(p) = \sqrt{p^0 + m}\begin{pmatrix} \chi_r \\ \frac{\sigma.\mathbf{p}}{p^0+m}\chi_r \end{pmatrix}$$

where $p^0 = \sqrt{\mathbf{p}^2 + m^2}$ and $\chi_{1,2}$ are orthonormal.

(a) Show that

$$\overline{u_r(p)}u_s(p) = 2m\delta_{rs}.$$

(b) Taking $\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, show that if we define P_+ by

$$P_+ = \frac{1}{2m}\left(u_1(p)\overline{u_1(p)} + u_2(p)\overline{u_2(p)}\right),$$

then we have

$$P_+ = \frac{1}{2m}(\gamma.p + m).$$

Show, either from the definition together with

$$\overline{u_r(p)}u_s(p) = 2m\delta_{rs},$$

or from the result above, that

$$P_+^2 = P_+.$$

Similarly, define

$$P_- = -\frac{1}{2m}\left(v_1(p)\overline{v_1(p)} + v_2(p)\overline{v_2(p)}\right),$$

where $v_r(p)$ is as defined in Qu. 3, Homework 6. With the same choice for $\chi_{1,2}$, show that

$$P_- = \frac{1}{2m}(-\gamma.p + m).$$

Notice that $P_+ + P_- = 1$. P_+ and P_- are **projection operators** onto the spaces of positive and negative energy states respectively.