MATH425 - Quantum Field Theory
Set Work: Sheet 3

1. The defining equation for the Lorentz group may be written

$$L^T \eta L = \eta. \qquad (1)$$

Consider a 2-dimensional spacetime for which $\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Show that the standard Lorentz transformation

$$L = \begin{pmatrix} \gamma & -\frac{\gamma v}{c} \\ -\frac{\gamma v}{c} & \gamma \end{pmatrix},$$

where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, satisfies the above condition.

2. We need to show that $\mathcal{L}_+^\uparrow$ is a group. This is done as follows:

(i) Use (1) to show that

$$L\eta^{-1}L^T = \eta^{-1}. \qquad (2)$$

(Hint: recall that for matrices $(AB)^{-1} = B^{-1}A^{-1}$.)

(ii) We now have from (1), (2)

$$\eta_{\alpha\beta}L^\alpha{}_\mu L^\beta{}_\nu = \eta_{\mu\nu}, \quad \eta^{\alpha\beta}L^\mu{}_\alpha L^\nu{}_\beta = \eta^{\mu\nu}. \qquad (3)$$

Let $\mathbf{l} = (L^1{}_0, L^2{}_0, L^3{}_0)$ and $\bar{\mathbf{l}} = (\bar{L}^0{}_1, \bar{L}^0{}_2, \bar{L}^0{}_3)$. By putting $\mu = \nu = 0$ in (3), show that $|\mathbf{l}| = \sqrt{(L^0{}_0)^2 - 1}$ and $|\bar{\mathbf{l}}| = \sqrt{(\bar{L}^0{}_0)^2 - 1}$.

(iii) By considering $(\bar{L}L)^0{}_0 = \bar{L}^0{}_\alpha L^\alpha{}_0$, show that

$$(\bar{L}L)^0{}_0 = \bar{L}^0{}_0 L^0{}_0 + \bar{\mathbf{l}}.\mathbf{l}.$$

(iv) Use the Schwartz inequality

$$|\bar{\mathbf{l}}.\mathbf{l}| \leq |\mathbf{l}||\bar{\mathbf{l}}|$$

to show

$$(\bar{L}L)^0{}_0 \geq \bar{L}^0{}_0 L^0{}_0 - \sqrt{(\bar{L}^0{}_0)^2 - 1}\sqrt{(L^0{}_0)^2 - 1}.$$

(v) Show that

$$(x-y)^2 \geq 0 \Rightarrow x^2y^2 - 2xy + 1 \geq (x^2-1)(y^2-1) \Rightarrow (xy-1)^2 \geq (x^2-1)(y^2-1)$$

$$\Rightarrow \text{either} \quad xy - 1 \geq \sqrt{x^2-1}\sqrt{y^2-1} \quad \text{or} \quad xy - 1 \leq -\sqrt{x^2-1}\sqrt{y^2-1}.$$

Deduce that if $x, y \geq 1$ then $xy - 1$ is positive and we must have

$$xy - \sqrt{x^2-1}\sqrt{y^2-1} \geq 1.$$

Finally combine with (iv) to deduce that if $\bar{L}^0{}_0 \geq 1$ and $L^0{}_0 \geq 1$, then $(\bar{L}L)^0{}_0 \geq 1$.

(vi) Use the fact that $\det(\bar{L}L) = \det\bar{L}\det L$ to deduce that

$$\det\bar{L} = \det L = 1 \Rightarrow \det(\bar{L}L) = 1.$$

(vii) We can now deduce that $L \in \mathcal{L}_+^\uparrow$ and $\bar{L} \in \mathcal{L}_+^\uparrow \Rightarrow (\bar{L}L) \in \mathcal{L}_+^\uparrow$. Together with the obvious fact that $1 \in \mathcal{L}_+^\uparrow$, this most of what we need to show that $\mathcal{L}_+^\uparrow$ is a group.

(viii) We still need to show that $L \in \mathcal{L}_+^\uparrow \Rightarrow L^{-1} \Rightarrow \mathcal{L}_+^\uparrow$. Note that (1) $\Rightarrow L^{-1} = \eta^{-1}L^T\eta$. So clearly $(L^{-1})^0{}_0 = L^0{}_0$. Moreover, $\det L^{-1} = \det\eta^{-1}\det L^T\det\eta = 1$. QED.