MATH425 - Quantum Field Theory
Set Work: Sheet 10

1. Using the normal-ordered energy-momentum vector for a scalar field,

$$P^\mu = \frac{1}{(2\pi)^3}\int d^3\mathbf{p}\frac{1}{2p^0}p^\mu[a^\dagger(\mathbf{p})a(\mathbf{p})],$$

and with ϕ given as usual by

$$\phi(x) = \frac{1}{(2\pi)^3}\int d^3\mathbf{p}\frac{1}{2p^0}\left[a(\mathbf{p})e^{-ip.x} + a^\dagger(\mathbf{p})e^{ip.x}\right],$$

show that

$$[P^\mu, \phi(x)] = -i\partial^\mu\phi(x).$$

Check that this implies that the equation

$$e^{-ia.P}\phi(x)e^{ia.P} = \phi(x-a)$$

is satisfied to 1st order in a. (In fact it is true to all orders.)

2. **(Revision)** Consider the expansion for $\phi(x)$ in real scalar quantum field theory

$$\phi(x) = \frac{1}{(2\pi)^3}\int d^3\mathbf{p}\frac{1}{2p^0}\left[a(\mathbf{p})e^{-ip.x} + a^\dagger(\mathbf{p})e^{ip.x}\right],$$

where $p^0 = \sqrt{\mathbf{p}^2 + m^2}$ and $a^\dagger(\mathbf{p})$ and $a(\mathbf{p})$ are creation and annihilation operators respectively for particles of momentum $\mathbf{p}$.

(i) Verify by direct computation that

$$a^\dagger(\mathbf{p}) = \int [p^0\phi(\mathbf{x},t) - i\pi(\mathbf{x},t)]e^{ip.x}d^3\mathbf{x},$$

where the 4-vector $x = (t, \mathbf{x})$ and t is arbitrary.

(ii) Assuming also the corresponding result

$$a(\mathbf{p}) = \int [p^0\phi(\mathbf{x},t) + i\pi(\mathbf{x},t)]e^{-ip.x}d^3\mathbf{x},$$

(which follows by hermitian conjugation) deduce the canonical commutation relations

$$\begin{aligned}[a(\mathbf{p}), a^\dagger(\mathbf{p}')] =& 2p^0\delta(\mathbf{p}-\mathbf{p}')(2\pi)^3,\\ [a^\dagger(\mathbf{p}), a^\dagger(\mathbf{p}')] =& 0.\end{aligned}$$

3. Show that

$$\phi^{(+)}(x_1)\phi^{(+)}(x_2)|\mathbf{p}_1\mathbf{p}_2> = e^{-i(p_1.x_1+p_2.x_2)}|0> + e^{-i(p_1.x_2+p_2.x_1)}|0>$$

4. Show that

$$<\mathbf{p}_3\mathbf{p}_4|\mathbf{p}_1\mathbf{p}_2> = <\mathbf{p}_4|\mathbf{p}_2><\mathbf{p}_3|\mathbf{p}_1> + <\mathbf{p}_4|\mathbf{p}_1><\mathbf{p}_3|\mathbf{p}_2>.$$

5. The Feynman propagator is defined by

$$\Delta_F(x) = \frac{1}{(2\pi)^4}\int_{\mathcal{C}_F} \frac{d^4k}{k^2 - m^2} e^{-ik.x}.$$

for an appropriate contour $\mathcal{C}_F$ in the complex k^0 plane.

(a) Sketch the contour $\mathcal{C}_F$.

(b) Defining

$$\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x)$$

where

$$\phi^{(+)}(x) = \frac{1}{(2\pi)^3}\int \frac{d^3\mathbf{p}}{2p^0} a(\mathbf{p}) e^{-ip.x}$$

$$\phi^{(-)}(x) = \frac{1}{(2\pi)^3}\int \frac{d^3\mathbf{p}}{2p^0} a^\dagger(\mathbf{p}) e^{ip.x},$$

show that for $x^0 > 0$

$$\Delta_F(x) = -i[\phi^{(+)}(x), \phi^{(-)}(0)],$$

while for $x^0 < 0$

$$\Delta_F(x) = -i[[\phi^{(+)}(0), \phi^{(-)}(x)].$$

(c) Hence show that

$$< 0|T\{\phi(x), \phi(0)\}|0 >= i\Delta_F(x).$$