MATH425 Quantum Field Theory Solutions 9

1.

$$
\begin{aligned}
Q = & : \int d^3\mathbf{x} \overline{\psi} \gamma^0 \psi :=: \int d^3\mathbf{x} \psi^\dagger \gamma^0 \gamma^0 \psi : \\
= & : \int d^3\mathbf{x} \psi^\dagger \psi :=:< \psi, \psi >: .
\end{aligned}
$$

Now

$$
\begin{aligned}
\psi(x) = & \frac{1}{(2\pi)^3} \sum_{s=1}^{2} \int \frac{d^3\mathbf{p}'}{2p'^0} [\psi_{p'}^{(s)}(x) a_s(\mathbf{p}') + \tilde{\psi}_{p'}^{(s)}(x) b_s^\dagger(\mathbf{p}')], \\
\psi^\dagger(x) = & \frac{1}{(2\pi)^3} \sum_{r=1}^{2} \int \frac{d^3\mathbf{p}}{2p^0} [\psi_{p}^{(r)\dagger}(x) a_r^\dagger(\mathbf{p}) + \tilde{\psi}_{p}^{(r)\dagger}(x) b_r(\mathbf{p})],
\end{aligned}
$$

where

$$
\begin{aligned}
\psi_p^{(s)}(x) = & e^{-ip.x} u_s(p), \\
\tilde{\psi}_p^{(s)}(x) = & e^{ip.x} v_s(p).
\end{aligned}
$$

So

$$
\begin{aligned}
Q = & \frac{1}{(2\pi)^3} \sum_{r=1}^{2} \int \frac{d^3\mathbf{p}}{2p^0} \frac{1}{(2\pi)^3} \sum_{s=1}^{2} \int \frac{d^3\mathbf{p}'}{2p'^0} \int d^3\mathbf{x} : [\psi_{p}^{(r)\dagger}(x) a_r^\dagger(\mathbf{p}) + \tilde{\psi}_{p}^{(r)\dagger}(x) b_r(\mathbf{p})] \\
& [\psi_{p'}^{(s)}(x) a_s(\mathbf{p}') + \tilde{\psi}_{p'}^{(s)}(x) b_s^\dagger(\mathbf{p}')] : \\
= & \frac{1}{(2\pi)^6} \sum_{r=1}^{2} \sum_{s=1}^{2} \int \frac{d^3\mathbf{p}}{2p^0} \int \frac{d^3\mathbf{p}'}{2p'^0} \int d^3\mathbf{x} \\
& : \Big[\psi_{p}^{(r)\dagger}(\mathbf{x}) \psi_{p'}^{(s)}(\mathbf{x}) a_r^\dagger(\mathbf{p}) a_s(\mathbf{p}') + \psi_{p}^{(r)\dagger}(\mathbf{x}) \tilde{\psi}_{p'}^{(s)}(\mathbf{x}) a_r^\dagger(\mathbf{p}) b_s^\dagger(\mathbf{p}') \\
& + \tilde{\psi}_{p}^{(r)\dagger}(\mathbf{x}) \psi_{p'}^{(s)}(\mathbf{x}) b_r(\mathbf{p}) a_s(\mathbf{p}') + \tilde{\psi}_{p}^{(r)\dagger}(\mathbf{x}) \psi_{p'}^{(s)}(\mathbf{x}) b_r(\mathbf{p}) b_s^\dagger(\mathbf{p}') \Big] : \\
= & \frac{1}{(2\pi)^6} \sum_{r=1}^{2} \sum_{s=1}^{2} \int \frac{d^3\mathbf{p}}{2p^0} \int \frac{d^3\mathbf{p}'}{2p'^0} \\
& : \Big[< \psi_p^{(r)}, \psi_{p'}^{(s)} > a_r^\dagger(\mathbf{p}) a_s(\mathbf{p}') + < \psi_p^{(r)}, \tilde{\psi}_{p'}^{(s)} > a_r^\dagger(\mathbf{p}) b_s^\dagger(\mathbf{p}') \\
& + < \tilde{\psi}_p^{(r)}, \psi_{p'}^{(s)} > b_r(\mathbf{p}) a_s(\mathbf{p}') + < \tilde{\psi}_p^{(r)}, \tilde{\psi}_{p'}^{(s)} > b_r(\mathbf{p}) b_s^\dagger(\mathbf{p}') \Big] : .
\end{aligned}
$$

We have

$$
\begin{aligned}
< \psi_p^{(r)}, \psi_{p'}^{(s)} > = < \tilde{\psi}_p^{(r)}, \tilde{\psi}_{p'}^{(s)} > = \delta_{rs} 2p^0 (2\pi)^3 \delta(\mathbf{p} - \mathbf{p}'), \\
< \psi_p^{(r)}, \tilde{\psi}_{p'}^{(s)} > = < \tilde{\psi}_p^{(r)}, \psi_{p'}^{(s)} > = 0.
\end{aligned}
$$

So

$$
\begin{aligned}
Q =& \frac{1}{(2\pi)^6}\sum_{r=1}^{2}\sum_{s=1}^{2}\int\frac{d^3\mathbf{p}}{2p^0}\int\frac{d^3\mathbf{p}'}{2p'^0}\delta_{rs}2p^0(2\pi)^3\delta(\mathbf{p}-\mathbf{p}') \\
& : \left[a_r^\dagger a_s(\mathbf{p}') + b_r(\mathbf{p})b_s^\dagger(\mathbf{p}')\right] : \\
=& \frac{1}{(2\pi)^3}\sum_{r=1}^{2}\int\frac{d^3\mathbf{p}}{2p^0} : \left[a_r^\dagger(\mathbf{p})a_r(\mathbf{p}) + b_r(\mathbf{p})b_r^\dagger(\mathbf{p})\right] : \\
=& \frac{1}{(2\pi)^3}\sum_{r=1}^{2}\int\frac{d^3\mathbf{p}}{2p^0} \left[a_r^\dagger(\mathbf{p})a_r(\mathbf{p}) - b_r^\dagger(\mathbf{p})b_r(\mathbf{p})\right]
\end{aligned}
$$

since $: b_r(\mathbf{p})b_r^\dagger(\mathbf{p}) := -b_r^\dagger(\mathbf{p})b_r(\mathbf{p})$. So

$$
Q = \int d^3\mathbf{p}[N_a(\mathbf{p}) - N_b(\mathbf{p})],
$$

where

$$
N_a(\mathbf{p}) = \frac{1}{(2\pi)^3}\frac{1}{2p^0}a_r^\dagger(\mathbf{p})a_r(\mathbf{p}).
$$

2(i). We write

$$
(\partial\phi)^2 = \eta_{\rho\sigma}\partial^\rho\phi\partial^\sigma\phi,
$$

then using

$$
\frac{\partial(\partial^\rho)\phi}{\partial(\partial^\mu\phi)} = \delta^\rho{}_\mu,
$$

we have

$$
\frac{\partial}{\partial(\partial^\mu\phi)}(\partial\phi)^2 = \eta_{\rho\sigma}(\delta^\rho{}_\mu\partial^\sigma\phi + \partial^\rho\phi\delta^\sigma{}_\mu) = 2\partial_\mu\phi.
$$

So equation of motion becomes

$$
\partial^\mu\left(\partial_\mu\phi\right) = -m^2\phi - \frac{1}{2}\lambda_3\phi^2 - \frac{1}{3!}\lambda_4\phi^3,
$$

or

$$
\partial^2\phi + m^2\phi + \frac{1}{2}\lambda_3\phi^2 + \frac{1}{3!}\lambda_4\phi^3 = 0.
$$

Generalised momentum $\pi = \dot{\phi}$ as in free case. So we have

$$
H = \int[\tfrac{1}{2}\dot{\phi}(x')^2 + \tfrac{1}{2}\left[\nabla\phi(x')\right]^2 + \tfrac{1}{2}m^2\phi(x')^2 + \frac{1}{3!}\lambda_3\phi(x')^3 + \frac{1}{4!}\lambda_4\phi(x')^4]d^3\mathbf{x}'.
$$

We have (suppressing t)

$$
\begin{aligned}
[\pi(\mathbf{x}),\phi(\mathbf{x}')^3] =& \phi(\mathbf{x}')[\pi(\mathbf{x}),\phi(\mathbf{x}')^2] + [\pi(\mathbf{x}),\phi(\mathbf{x}')]\phi(\mathbf{x}')^2 \\
=& \phi(\mathbf{x}')\left(\phi(\mathbf{x}')[\pi(\mathbf{x}),\phi(\mathbf{x}')] + [\pi(\mathbf{x}),\phi(\mathbf{x}')]\phi(\mathbf{x}')\right) + [\pi(\mathbf{x}),\phi(\mathbf{x}')]\phi(\mathbf{x}')^2 \\
=& \phi(\mathbf{x}')^2[\pi(\mathbf{x}),\phi(\mathbf{x}')] + \phi(\mathbf{x}')[\pi(\mathbf{x}),\phi(\mathbf{x}')]\phi(\mathbf{x}') + [\pi(\mathbf{x}),\phi(\mathbf{x}')]\phi(\mathbf{x}') \\
=& -3\phi(\mathbf{x}')^2 i\hbar\delta(\mathbf{x}-\mathbf{x}').
\end{aligned}
$$

Similarly,

$$[\pi(\mathbf{x}), \phi(\mathbf{x}')^4] = -4\phi(\mathbf{x}')^3 i\hbar\delta(\mathbf{x}-\mathbf{x}').$$

Writing $H = H_0 + H_I$ where

$$H_0 = \int [\tfrac{1}{2}\dot{\phi}(x')^2 + \tfrac{1}{2}\left[\nabla\phi(x')\right]^2 + \tfrac{1}{2}m^2\phi(x')^2] d^3\mathbf{x}'$$
$$H_I = \int [\frac{1}{3!}\lambda_3\phi(x')^3 + \frac{1}{4!}\lambda_4\phi(x')^4] d^3\mathbf{x}',$$

we showed already (in the notes) that

$$[\pi(\mathbf{x}), H_0] = i\hbar\left[\nabla^2\phi(\mathbf{x}) - m^2\phi(\mathbf{x})\right].$$

We now have

$$\begin{aligned}[\pi(\mathbf{x}), H_I] &= -i\hbar\int\left[\lambda_3\frac{1}{2}\delta(\mathbf{x}-\mathbf{x}')\phi(\mathbf{x}')^2 + \lambda_4\frac{1}{3!}\delta(\mathbf{x}-\mathbf{x}')\phi(\mathbf{x}')^3\right] d^3\mathbf{x}' \\ &= -i\hbar\left[\frac{1}{2}\lambda_3\phi(\mathbf{x}')^2 + \frac{1}{3!}\lambda_4\phi(\mathbf{x}')^3.\right]\end{aligned}$$

So

$$[\pi(\mathbf{x}), H] = i\hbar\left[\nabla^2\phi(\mathbf{x}) - m^2\phi(\mathbf{x}) - \frac{1}{2}\lambda_3\phi(\mathbf{x}')^2 - \frac{1}{3!}\lambda_4\phi(\mathbf{x}')^3\right]$$

and

$$i\hbar\dot{\pi} = [\pi, H]$$

gives

$$\dot{\pi} = \ddot{\phi} = \nabla^2\phi(\mathbf{x}) - m^2\phi(\mathbf{x}) - \frac{1}{2}\lambda_3\phi(\mathbf{x}')^2 - \frac{1}{3!}\lambda_4\phi(\mathbf{x}')^3,$$

which (since $\partial^2\phi = \ddot{\phi} - \nabla^2\phi$) implies

$$\partial^2\phi + m^2\phi + \frac{1}{2}\lambda_3\phi^2 + \frac{1}{3!}\lambda_4\phi^3 = 0.$$

as required.