MATH425 Quantum Field Theory Solutions 8

1(a). We have

$$\begin{aligned}
<\psi_p^{(r)},\tilde{\psi}_{p'}^{(s)}>=&\int d^3\mathbf{x}\sqrt{p^0+m}\sqrt{p'^0+m}e^{ip.x}e^{ip'.x}\begin{pmatrix}\chi_r^\dagger & \chi_r^\dagger\frac{\sigma.\mathbf{p}}{p^0+m}\end{pmatrix}\begin{pmatrix}\frac{\sigma.\mathbf{p}'}{p^0+m}\chi_s\\ \chi_s\end{pmatrix}\\
=&\int d^3\mathbf{x}\sqrt{p^0+m}\sqrt{p'^0+m}e^{i(p^0+p'^0)x^0}e^{-i(\mathbf{p}+\mathbf{p}').\mathbf{x}}\\
&\left(\chi_r^\dagger\frac{\sigma.\mathbf{p}'}{p^0+m}\chi_s+\chi_r^\dagger\frac{\sigma.\mathbf{p}}{p^0+m}\chi_s\right)\\
=&\sqrt{p^0+m}\sqrt{p'^0+m}e^{i(p^0+p'^0)x^0}(2\pi)^3\delta(\mathbf{p}+\mathbf{p}')\\
&\left(\chi_r^\dagger\frac{\sigma.\mathbf{p}'}{p^0+m}\chi_s+\chi_r^\dagger\frac{\sigma.\mathbf{p}}{p^0+m}\chi_s\right)\\
=&(2\pi)^3(p^0+m)e^{2ip^0}\left(-\chi_r^\dagger\frac{\sigma.\mathbf{p}}{p^0+m}\chi_s+\chi_r^\dagger\frac{\sigma.\mathbf{p}}{p^0+m}\chi_s\right)\\
=&0.
\end{aligned}$$

Taking the hermitian conjugate of this result, we also have

$$<\tilde{\psi}_p^{(r)},\psi_{p'}^{(s)}>=0,$$

and we can also derive (in the same way as for $<\psi_p^{(r)},\psi_{p'}^{(s)}>$)

$$<\tilde{\psi}_p^{(r)},\tilde{\psi}_{p'}^{(s)}>=\delta_{rs}2p^0(2\pi)^3\delta(\mathbf{p}-\mathbf{p}').$$

Taking the scalar product of $\tilde{\psi}_p^{(s)}$ with

$$\psi(x)=\frac{1}{(2\pi)^3}\sum_{r=1}^{2}\int\frac{d^3\mathbf{p}'}{2p'^0}[\psi_{p'}^{(r)}(x)a_r(\mathbf{p}')+\tilde{\psi}_{p'}^{(r)}(x)b_r^\dagger(\mathbf{p}')],$$

we find

$$\begin{aligned}
<\tilde{\psi}_p^{(s)},\psi>=&\frac{1}{(2\pi)^3}\sum_{r=1}^{2}\int\frac{d^3\mathbf{p}'}{2p'^0}[<\tilde{\psi}_p^{(s)},\psi_{p'}^{(r)}>a_r(\mathbf{p}')+<\tilde{\psi}_p^{(s)},\tilde{\psi}_{p'}^{(r)}>b_r^\dagger(\mathbf{p}')]\\
=&\frac{1}{(2\pi)^3}\sum_{r=1}^{2}\int\frac{d^3\mathbf{p}'}{2p'^0}\delta_{rs}2p^0(2\pi)^3\delta(\mathbf{p}-\mathbf{p}')b_r^\dagger(\mathbf{p}')\\
=&b_s^\dagger(\mathbf{p}),
\end{aligned}$$

$$\text{i.e.}\quad b_s^\dagger(\mathbf{p}')=\int d^3\mathbf{x}'\tilde{\psi}_{p'\beta}^{(s)\dagger}(x')\psi_\beta(x'),$$

and therefore

$$b_r(\mathbf{p})=<\psi,\tilde{\psi}_p^{(r)}>=\int d^3\mathbf{x}\psi_\alpha^\dagger(x)\tilde{\psi}_{p\alpha}^{(r)}(x).$$

So

$$
\begin{aligned}
\{b_r(\mathbf{p}), b_s^\dagger(\mathbf{p'})\} &= \int d^3\mathbf{x} d^3\mathbf{x'} \tilde{\psi}^{(s)\dagger}_{p'\beta}(x')\tilde{\psi}^{(r)}_{p\alpha}(x)\{\psi_\beta(x'), \psi^\dagger_\alpha(x)\} \\
&= \int d^3\mathbf{x} d^3\mathbf{x'} \tilde{\psi}^{(s)\dagger}_{p'\beta}(x')\tilde{\psi}^{(r)}_{p\alpha}(x)\delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{x'}) \\
&= \int d^3\mathbf{x} \tilde{\psi}^{(s)\dagger}_{p'\alpha}(x)\tilde{\psi}^{(r)}_{p\alpha}(x) \\
&= < \tilde{\psi}^{(s)}_{p'}, \tilde{\psi}^{(r)}_{p} > \\
&=\delta_{rs} 2p^0 (2\pi)^3 \delta(\mathbf{p}-\mathbf{p'}).
\end{aligned}
$$

(b) I'll give here the correct derivation, which requires a slight (but crucial) modification in the definition of P^μ.

$$
\begin{aligned}
P^\mu =& \frac{i}{2}\int d^3\mathbf{x}\overline{\psi}\gamma^\mu\partial^0\psi \\
=& \frac{i}{2}\frac{1}{(2\pi)^3}\sum_{r=1}^{2}\int d^3\mathbf{x}\int\frac{d^3\mathbf{p}}{2p^0}\left\{\overline{u}_r(p)e^{ip.x}a_r^\dagger(\mathbf{p}) + \overline{v}_r(p)e^{-ip.x}b_r(\mathbf{p})\right\}\gamma^\mu \\
& \frac{1}{(2\pi)^3}\sum_{s=1}^{2}\int\frac{d^3\mathbf{p'}}{2p'^0}\left\{(-ip'^0 u_s(p')e^{-ip'.x}a_s(\mathbf{p'}) + ip'^0 v_s(p')e^{ip'.x}b_s^\dagger(\mathbf{p'})\right\} \\
& -\frac{i}{2}\frac{1}{(2\pi)^3}\sum_{r=1}^{2}\int d^3\mathbf{x}\int\frac{d^3\mathbf{p}}{2p^0}\left\{ip^0\overline{u}_r(p)e^{ip.x}a_r^\dagger(\mathbf{p}) + (-ip^0)\overline{v}_r(p)e^{-ip.x}b_r(\mathbf{p})\right\}\gamma^\mu \\
& \frac{1}{(2\pi)^3}\sum_{s=1}^{2}\int\frac{d^3\mathbf{p'}}{2p'^0}\left\{u_s(p')e^{-ip'.x}a_s(\mathbf{p'}) + v_s(p')e^{ip'.x}b_s^\dagger(\mathbf{p'})\right\} \\
=& \frac{1}{2}\frac{1}{(2\pi)^6}\sum_{r=1}^{2}\sum_{s=1}^{2}\int d^3\mathbf{x}\int\frac{d^3\mathbf{p}}{2p^0}\int\frac{d^3\mathbf{p'}}{2p'^0} \\
& \Big[(p^0+p'^0)e^{i(p-p').x}\overline{u}_r(p)\gamma^\mu u_s(p')a_r^\dagger(\mathbf{p})a_s(\mathbf{p'}) \\
& + (p^0-p'^0)e^{i(p+p').x}\overline{u}_r(p)\gamma^\mu v_s(p')a_r^\dagger(\mathbf{p})b_s^\dagger(\mathbf{p'}) \\
& + (-p^0+p'^0)e^{i(-p-p').x}\overline{v}_r(p)\gamma^\mu u_s(p')b_r(\mathbf{p})a_s(\mathbf{p'}) \\
& + (-p^0-p'^0)e^{i(p'-p).x}\overline{v}_r(p)\gamma^\mu v_s(p')b_r(\mathbf{p})b_s^\dagger(\mathbf{p'})\Big]
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{1}{2}\frac{1}{(2\pi)^6}\sum_{r=1}^{2}\sum_{s=1}^{2}\int\frac{d^3\mathbf{p}}{2p^0}\int\frac{d^3\mathbf{p}'}{2p'^0}\\
&\Big[(p^0+p'^0)e^{i(p^0-p'^0)x^0}(2\pi)^3\delta(\mathbf{p}-\mathbf{p}')\overline{u}_r(p)\gamma^\mu u_s(p')a_r^\dagger(\mathbf{p})a_s(\mathbf{p}')\\
&+(p^0-p'^0)e^{i(p^0+p'^0)x^0}(2\pi)^3\delta(\mathbf{p}+\mathbf{p}')\overline{u}_r(p)\gamma^\mu v_s(p')a_r^\dagger(\mathbf{p})b_s^\dagger(\mathbf{p}')\\
&+(-p^0+p'^0)e^{i(-p^0-p'^0)x^0}(2\pi)^3\delta(\mathbf{p}+\mathbf{p}')\overline{v}_r(p)\gamma^\mu u_s(p')b_r(\mathbf{p})a_s(\mathbf{p}')\\
&+(-p^0-p'^0)e^{i(p'^0-p^0)x^0}(2\pi)^3\delta(\mathbf{p}-\mathbf{p}')\overline{v}_r(p)\gamma^\mu v_s(p')b_r(\mathbf{p})b_s^\dagger(\mathbf{p}')\Big]\\
=&\frac{1}{2}\frac{1}{(2\pi)^3}\sum_{r=1}^{2}\sum_{s=1}^{2}\int\frac{d^3\mathbf{p}}{2p^0}\\
&\frac{1}{2p^0}\left[2p^0\overline{u}_r(p)\gamma^\mu u_s(p)a_r^\dagger(\mathbf{p})a_s(\mathbf{p})-2p^0\overline{v}_r(p)\gamma^\mu v_s(p)b_r(\mathbf{p})b_s^\dagger(\mathbf{p})\right]\\
=&\frac{1}{2}\frac{1}{(2\pi)^3}\sum_{r=1}^{2}\sum_{s=1}^{2}\int\frac{d^3\mathbf{p}}{2p^0}\\
&\left[2p^\mu\delta_{rs}a_r^\dagger(\mathbf{p})a_s(\mathbf{p})-2p^\mu\delta_{rs}b_r(\mathbf{p})b_s^\dagger(\mathbf{p})\right]\\
=&\frac{1}{(2\pi)^3}\sum_{r=1}^{2}\int\frac{d^3\mathbf{p}}{2p^0}p^\mu\left[a_r^\dagger(\mathbf{p})a_r(\mathbf{p})-b_r(\mathbf{p})b_r^\dagger(\mathbf{p})\right]
\end{aligned}
$$

Proof that $\overline{u}_r(\mathbf{p})\gamma^\mu u_s(\mathbf{p})=2p^\mu\delta_{rs}$. ∎

$$
\begin{aligned}
\overline{u}_r(\mathbf{p})\gamma^i u_s(\mathbf{p})=&(p^0+m)\begin{pmatrix}\chi_r^\dagger & \chi_r^\dagger\frac{\sigma.\mathbf{p}}{p^0+m}\end{pmatrix}\gamma^0\gamma^i\begin{pmatrix}\chi_s\\ \frac{\sigma.\mathbf{p}}{p^0+m}\chi_s\end{pmatrix}\\
=&(p^0+m)\begin{pmatrix}\chi_r^\dagger & \chi_r^\dagger\frac{\sigma.\mathbf{p}}{p^0+m}\end{pmatrix}\begin{pmatrix}0_2 & \sigma_i\\ \sigma_i & 0_2\end{pmatrix}\begin{pmatrix}\chi_s\\ \frac{\sigma.\mathbf{p}}{p^0+m}\chi_s\end{pmatrix}\\
=&(p^0+m)\chi_r^\dagger\left(\sigma^i\frac{\sigma.\mathbf{p}}{p^0+m}+\frac{\sigma.\mathbf{p}}{p^0+m}\sigma^i\right)\chi_s\\
=&p^j\chi_r^\dagger\left(\sigma^i\sigma^j+\sigma^j\sigma^i\right)\chi_s\\
=&p^j 2\delta_{ij}\chi_r^\dagger\chi_s\\
=&2p^i\delta_{rs}.
\end{aligned}
$$

$$
\begin{aligned}
\overline{u}_r(\mathbf{p})\gamma^0 u_s(\mathbf{p}) =& (p^0+m)\begin{pmatrix} \chi_r^\dagger & \chi_r^\dagger \frac{\sigma.\mathbf{p}}{p^0+m} \end{pmatrix} (\gamma^0)^2 \begin{pmatrix} \chi_s \\ \frac{\sigma.\mathbf{p}}{p^0+m}\chi_s \end{pmatrix} \\
=& (p^0+m)\begin{pmatrix} \chi_r^\dagger & \chi_r^\dagger \frac{\sigma.\mathbf{p}}{p^0+m} \end{pmatrix} \begin{pmatrix} \chi_s \\ \frac{\sigma.\mathbf{p}}{p^0+m}\chi_s \end{pmatrix} \\
=& (p^0+m)\chi_r^\dagger \left[1 + \left(\frac{\sigma.\mathbf{p}}{p^0+m}\right)^2\right]\chi_s \\
=& (p^0+m)\chi_r^\dagger \left[1 + \frac{\mathbf{p}^2}{(p^0+m)^2}\right]\chi_s \\
=& \left[p^0 + m + \frac{(p^0)^2 - m^2}{p^0+m}\right]\delta_{rs} \\
=& (p^0 + m + p^0 - m)\delta_{rs} = 2p^0\delta_{rs} \\
\text{So}\quad \overline{u}_r(\mathbf{p})\gamma^\mu u_s(\mathbf{p}) =& 2p^\mu \delta_{rs}
\end{aligned}
$$