MATH425 Quantum Field Theory Solutions 7

1.

$$\begin{aligned}
\alpha_H =& U^\dagger \alpha U \\
i\hbar \frac{\partial \alpha_H}{\partial t} =& i\hbar \frac{\partial U^\dagger}{\partial t} \alpha U + i\hbar U^\dagger \alpha \frac{\partial U}{\partial t} \\
=& -\left(i\hbar \frac{\partial U}{\partial t} \right)^\dagger \alpha U + U^\dagger \alpha \left(i\hbar \frac{\partial U}{\partial t} \right) \\
=& -(HU)^\dagger \alpha U + U^\dagger \alpha H U \\
=& -U^\dagger H \alpha U + U^\dagger \alpha H U \quad (H^\dagger = H) \\
=& U^\dagger \alpha U U^\dagger H U - U^\dagger H U U^\dagger \alpha U \\
=& \alpha_H H_H - H_H \alpha_H = [\alpha_H, H_H].
\end{aligned}$$

2. Two-particle states are defined by

$$\begin{aligned}
|\mathbf{p}_1, \mathbf{p}_2 >=& a^\dagger(\mathbf{p}_1) a^\dagger(\mathbf{p}_1)|0> \\
|\mathbf{p}_1, \mathbf{p}_2 >=& |\mathbf{p}_2, \mathbf{p}_1 > \quad \text{as} \quad [a(\mathbf{p}_1), a(\mathbf{p}_2)] = 0 \\
< \mathbf{p}_1', \mathbf{p}_2' | \mathbf{p}_1, \mathbf{p}_2 >=& < 0| a(\mathbf{p}_1') a(\mathbf{p}_2') a^\dagger(\mathbf{p}_1) a^\dagger(\mathbf{p}_2)|0> \\
=& < 0| a(\mathbf{p}_1') \{ a^\dagger(\mathbf{p}_1) a(\mathbf{p}_2') + (2\pi)^3 2p_1^0 \delta(\mathbf{p}_1 - \mathbf{p}_2') \} a^\dagger(\mathbf{p}_2)|0> \\
=& < 0| a(\mathbf{p}_1') a^\dagger(\mathbf{p}_1) \{ a^\dagger(\mathbf{p}_2) a(\mathbf{p}_2') + (2\pi)^3 2p_1^0 \delta(\mathbf{p}_2 - \mathbf{p}_2') \} |0> \\
& + (2\pi)^3 2p_1^0 \delta(\mathbf{p}_1 - \mathbf{p}_2') < 0| a(\mathbf{p}_1') a^\dagger(\mathbf{p}_2)|0> \\
=& (2\pi)^3 2p_2^0 \delta(\mathbf{p}_2 - \mathbf{p}_2') < 0| \{ a^\dagger(\mathbf{p}_1) a(\mathbf{p}_1') + (2\pi)^3 2p_1^0 \delta(\mathbf{p}_1 - \mathbf{p}_1') \} |0> \\
& + (2\pi)^3 2p_1^0 \delta(\mathbf{p}_1 - \mathbf{p}_2') < 0| \{ a^\dagger(\mathbf{p}_2) a(\mathbf{p}_1') + (2\pi)^3 2p_2^0 \delta(\mathbf{p}_2 - \mathbf{p}_1') \} |0> \\
=& (2\pi)^6 (2p_1^0)(2p_2^0) \{ \delta(\mathbf{p}_1 - \mathbf{p}_1') \delta(\mathbf{p}_2 - \mathbf{p}_2') + \delta(\mathbf{p}_1 - \mathbf{p}_2') \delta(\mathbf{p}_2 - \mathbf{p}_1') \}.
\end{aligned}$$

3.

$$\begin{aligned}
\int |\tilde{\psi}(\mathbf{p})|^2 d^3\mathbf{p} =& \frac{1}{(2\pi)^3} \int d^3\mathbf{p} d^3\mathbf{x} d^3\mathbf{x}' \psi^*(\mathbf{x}) \psi(\mathbf{x}') e^{-i\mathbf{p}.\mathbf{x}} e^{i\mathbf{p}.\mathbf{x}'} \\
=& \frac{1}{(2\pi)^3} \int d^3\mathbf{p} d^3\mathbf{x} d^3\mathbf{x}' \psi^*(\mathbf{x}) \psi(\mathbf{x}') e^{i\mathbf{p}.(\mathbf{x}' - \mathbf{x})} \\
=& \frac{1}{(2\pi)^3} \int d^3\mathbf{x} d^3\mathbf{x}' \psi^*(\mathbf{x}) \psi(\mathbf{x}') (2\pi)^3 \delta(\mathbf{x} - \mathbf{x}') \\
=& \int d^3\mathbf{x} \psi^*(\mathbf{x}) \psi(\mathbf{x}) = \int d^3\mathbf{x} |\psi(\mathbf{x})|^2 = 1.
\end{aligned}$$

4.

$$
\begin{aligned}
N =& \frac{1}{(2\pi)^3}\int \frac{d^3\mathbf{p}'}{2p'^0} a^\dagger(\mathbf{p}')a(\mathbf{p}') \\
\Rightarrow [N, a^\dagger(\mathbf{p})] =& \left[\frac{1}{(2\pi)^3}\int \frac{d^3\mathbf{p}'}{2p'^0} a^\dagger(\mathbf{p}')a(\mathbf{p}'), a^\dagger(\mathbf{p})\right] \\
=& \frac{1}{(2\pi)^3}\int \frac{d^3\mathbf{p}'}{2p'^0} [a^\dagger(\mathbf{p}')a(\mathbf{p}'), a^\dagger(\mathbf{p})] \\
=& \frac{1}{(2\pi)^3}\int \frac{d^3\mathbf{p}'}{2p'^0} \left\{a^\dagger(\mathbf{p}')[a(\mathbf{p}'), a^\dagger(\mathbf{p})] + [a^\dagger(\mathbf{p}'), a^\dagger(\mathbf{p})]a(\mathbf{p}')\right\} \\
=& \frac{1}{(2\pi)^3}\int \frac{d^3\mathbf{p}'}{2p'^0} a^\dagger(\mathbf{p}')2p'^0(2\pi)^3\delta(\mathbf{p}-\mathbf{p}') \\
=& a^\dagger(\mathbf{p}).
\end{aligned}
$$

So we have

$$
\begin{aligned}
Na^\dagger(\mathbf{p}) - a^\dagger(\mathbf{p})N =& a^\dagger(\mathbf{p}) \\
\Rightarrow Na^\dagger(\mathbf{p}) =& a^\dagger(\mathbf{p})(N+1) \\
\Rightarrow N|\mathbf{p}_1 \ldots \mathbf{p}_n > =& Na^\dagger(\mathbf{p}_1)\ldots a^\dagger(\mathbf{p}_n)|0> \\
=& a^\dagger(\mathbf{p}_1)(N+1)a^\dagger(\mathbf{p}_2)\ldots a^\dagger(\mathbf{p}_n)|0> \\
=& a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)(N+2)\ldots a^\dagger(\mathbf{p}_n)|0> \\
=& \ldots = a^\dagger(\mathbf{p}_1)\ldots a^\dagger(\mathbf{p}_n)(N+n)|0> \\
=& na^\dagger(\mathbf{p}_1)\ldots a^\dagger(\mathbf{p}_n)|0> \quad (a(\mathbf{p})|0>=0) \\
=& n|\mathbf{p}_1 \ldots \mathbf{p}_n > .
\end{aligned}
$$

4(a). Under a Lorentz transformation we have

$$
\begin{aligned}
j'^{\mu}(x') =&\overline{\psi'(x')}\gamma^5\gamma^\mu\psi'(x') = \overline{\psi(x)}S(L)^{-1}\gamma^5\gamma^\mu S(L)\psi(x)\\
=&\overline{\psi(x)}S(L)^{-1}\gamma^5 S(L)S(L)^{-1}\gamma^\mu S(L)\psi(x) = L^\mu{}_\nu\overline{\psi(x)}\gamma^5\gamma^\nu\psi(x) = L^\mu{}_\nu j^\nu(x),
\end{aligned}
$$

using $S(L)^{-1}\gamma^5 S(L) = \gamma^5$, $S(L)^{-1}\gamma^\mu S(L) = L^\mu{}_\nu\gamma^\nu$. Under a parity transformation we have

$$
\begin{aligned}
j'^{\mu}(x') =&\overline{\psi'(x')}\gamma^5\gamma^\mu\psi'(x') = \overline{\psi(x)}S(P)^{-1}\gamma^5\gamma^\mu S(P)\psi(x)\\
=&\overline{\psi(x)}S(P)^{-1}\gamma^5 S(P)S(P)^{-1}\gamma^\mu S(P)\psi(x)\\
=&-P^\mu{}_\nu\overline{\psi(x)}\gamma^5\gamma^\nu\psi(x) = -P^\mu{}_\nu j^\nu(x),
\end{aligned}
$$

using $S(P)^{-1}\gamma^5 S(P) = -\gamma^5$, $S(P)^{-1}\gamma^\mu S(P) = P^\mu{}_\nu\gamma^\nu$.

(4b) Consider taking for L a small rotation through an angle θ about an axis $\mathbf{n}$. Then

$$
\begin{aligned}
L^\mu{}_0 = L^0{}_\mu =&0\\
\delta\mathbf{x} =&-\mathbf{x}\times\mathbf{n}\theta\\
\Rightarrow L^i{}_j =&x^i - \epsilon_{ijk}x^j n^k\theta\\
\epsilon^i{}_j =&-\epsilon_{ijk}n^k\theta = -\epsilon^{ij}\\
\text{So}\quad S(L) =&1 - \frac{i}{4}\epsilon_{ijk}n^k\theta\sigma^{ij}\\
=&1 + \frac{1}{8}\epsilon_{ijk}n^k[\gamma^i,\gamma^j]\theta.
\end{aligned}
$$

$$
\text{Now}\quad [\gamma^i,\gamma^j] = -2i\epsilon_{ijl}\begin{pmatrix}\sigma^l & 0\\ 0 & \sigma^l\end{pmatrix}
$$

in the given representation, so (using $\epsilon_{ijk}\epsilon_{ijl} = 2\delta_{kl}$)

$$
\begin{aligned}
S(L) =&1 - \frac{i}{2}n^k\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix}\theta\\
=&1 - i\theta\mathbf{n}.\mathbf{S},\\
\text{where}\quad S^k =&\frac{1}{2}\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix},\\
\text{and since}\quad [\tfrac{1}{2}\sigma^i, \tfrac{1}{2}\sigma^j] =&i\epsilon_{ijk}\tfrac{1}{2}\sigma^k,\\
\text{we have}\quad [S^i, S^j] =& i\epsilon_{ijk}S^k.
\end{aligned}
$$

Consider a scalar field $\phi(x)$. In a Lorentz-transformed frame we have

$$
\phi'(x') = \phi(x) = \phi(L^{-1}x'),\quad \text{or equivalently}\quad \phi'(x) = \phi(L^{-1}x).
$$

We have

$$\begin{aligned}
(Lx)^i &= x^i - \epsilon_{ijk} x^j n^k \theta, \\
\text{so} \quad (L^{-1}x)^i &= x^i + \epsilon_{ijk} x^j n^k \theta, \\
\text{and so} \quad \phi'(x) &= \phi(x^i + \epsilon_{ijk} x^j n^k \theta) \\
&= \phi(x^i) + \epsilon_{ijk} x^j n^k \theta \frac{\partial \phi}{\partial x^i} + O(\theta^2) \\
&= \phi(x) - \theta \mathbf{n}.(\mathbf{x} \times \nabla)\phi \\
&= \phi(x) - i\theta \mathbf{n}.(\mathbf{x} \times \mathbf{p})\phi \\
&= (1 - i\theta \mathbf{n}.\mathbf{L})\phi(x).
\end{aligned}$$

$$\text{So} \quad \delta\phi = -i\theta\mathbf{n}.\mathbf{L}\phi,$$

i.e. the generator of the change in ϕ corresponding to a rotation is the component of $\mathbf{L}$ in that direction. Now for the Dirac field $\psi(x)$,

$$\begin{aligned}
\psi'(x) &= S(L)\psi(L^1 x) \\
&= (1 - i\theta\mathbf{n}.\mathbf{S})(1 - i\theta\mathbf{n}.\mathbf{L})\psi(x) \\
&= (1 - i\theta\mathbf{n}.\mathbf{J})\psi(x) \quad \text{where} \quad \mathbf{J} = \mathbf{L} + \mathbf{S}.
\end{aligned}$$