MATH425 Quantum Field Theory Solutions 6

1. Consider the plane wave $\psi(x) = v(p)e^{ip.x}$. It will satisfy the Dirac equation

$$\begin{aligned} (i\gamma.\partial - m)\psi =& 0 \\ \text{if} \quad (\gamma.p + m)v(p) =& 0. \end{aligned}$$

In our representation,

$$\gamma.p + m = \begin{pmatrix} p^0 + m & -\sigma.\mathbf{p} \\ \sigma.\mathbf{p} & -p^0 + m \end{pmatrix}$$

where each entry is a 2×2 block. Now write

$$v(p) = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \quad \text{where} \quad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \quad \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}.$$

Then

$$(\gamma.p + m)v(p) = 0$$

becomes

$$\begin{aligned} (p^0 + m)\xi =& \sigma.\mathbf{p}\eta, \\ \sigma.\mathbf{p}\xi =& (p^0 - m)\eta. \end{aligned}$$

Given η, define

$$\begin{aligned} \xi =& \frac{\sigma.\mathbf{p}}{p^0 + m}\eta \\ \text{then} \quad \sigma.\mathbf{p}\xi =& \frac{(\sigma.\mathbf{p})^2}{p^0 + m}\eta \\ \text{Now} \quad \sigma^i\sigma^j p_i p_j =& \mathbf{p}^2 \quad (\sigma^i\sigma^j = \delta_{ij} + i\epsilon_{ijk}\sigma^k) \\ \text{So} \quad \sigma.\mathbf{p}\xi =& \frac{\mathbf{p}^2}{p^0 + m}\eta \\ =& \frac{(p^0)^2 - m^2}{p^0 + m}\eta \\ =& (p^0 - m)\eta \end{aligned}$$

so we have a consistent solution. The general form of $v(p)$ is

$$v(p) = N(p)\begin{pmatrix} \frac{\sigma.\mathbf{p}}{p^0+m}\chi \\ \chi \end{pmatrix},$$

where χ is a 2-vector and $N(p)$ is arbitrary. Suppose we pick two orthonormal 2-vectors $\chi_{1,2}$ satisfying

$$\chi_r^\dagger \chi_s = \delta_{rs}.$$

We have

$$
\begin{aligned}
\overline{v_r(p)}v_s(p) = v_r^\dagger(p)\gamma^0 v_s(p) =& N(p)^* N(p) \begin{pmatrix} \chi_r^\dagger \frac{\sigma.\mathbf{p}}{p^0+m} & \chi_r^\dagger \end{pmatrix} \gamma^0 \begin{pmatrix} \frac{\sigma.\mathbf{p}}{p^0+m}\chi_s \\ \chi_s \end{pmatrix} \\
=& N(p)^* N(p) \begin{pmatrix} \chi_r^\dagger \frac{\sigma.\mathbf{p}}{p^0+m} & \chi_r^\dagger \end{pmatrix} \begin{pmatrix} 1_2 & 0_2 \\ 0_2 & -1_2 \end{pmatrix} \begin{pmatrix} \frac{\sigma.\mathbf{p}}{p^0+m}\chi_s \\ \chi_s \end{pmatrix} \\
=& |N(p)|^2 \left[\chi_r^\dagger \frac{(\sigma.\mathbf{p})^2}{(p^0+m)^2}\chi_s - \chi_r^\dagger \chi_s \right] \\
=& |N(p)|^2 \left[\chi_r^\dagger \frac{\mathbf{p}^2}{(p^0+m)^2}\chi_s - \chi_r^\dagger \chi_s \right] \\
=& |N(p)|^2 \chi_r^\dagger \chi_s \frac{\mathbf{p}^2 - (p^0+m)^2}{(p^0+m)^2} \\
=& |N(p)|^2 \delta_{rs} \frac{(p^0)^2 - m^2 - (p^0+m)^2}{(p^0+m)^2} \\
=& |N(p)|^2 \delta_{rs} \frac{-2p^0 m - 2m^2}{(p^0+m)^2} = -2m\delta_{rs}
\end{aligned}
$$

if we take $N(p) = \sqrt{p^0+m}$.

2(a).

$$
\begin{aligned}
u_r(p) =& \sqrt{p^0+m}\begin{pmatrix} \chi_r \\ \frac{\sigma.\mathbf{p}}{p^0+m}\chi_r \end{pmatrix} \\
\Rightarrow \overline{u_r(p)}u_s(p) =& u_r^\dagger \gamma^0 u_s = (p^0+m)\begin{pmatrix} \chi_r^\dagger & \chi_r^\dagger \frac{\sigma.\mathbf{p}}{p^0+m} \end{pmatrix}\begin{pmatrix} 1_2 & 0_2 \\ 0_2 & -1_2 \end{pmatrix}\begin{pmatrix} \chi_s \\ \frac{\sigma.\mathbf{p}}{p^0+m}\chi_s \end{pmatrix} \\
=& (p^0+m)\left(\chi_r^\dagger\chi_s - \chi_r^\dagger \frac{(\sigma.\mathbf{p})^2}{(p^0+m)^2}\chi_s\right) \\
=& (p^0+m)\left(\chi_r^\dagger\chi_s - \chi_r^\dagger \frac{\mathbf{p}^2}{(p^0+m)^2}\chi_s\right) \\
=& (p^0+m)\left(1 - \frac{\mathbf{p}^2}{(p^0+m)^2}\right)\chi_r^\dagger\chi_s \\
=& (p^0+m)\left(\frac{(p^0+m)^2 - ((p^0)^2 - m^2)}{(p^0+m)^2}\right)\delta_{rs} \\
=& (p^0+m)\frac{2p^0m + 2m^2}{(p^0+m)^2}\delta_{rs} = 2m\delta_{rs}.
\end{aligned}
$$

(b).

$$
\begin{aligned}
u_1(p)\overline{u_1(p)} =& u_1(p)u_1^\dagger\gamma^0 = (p^0+m)\begin{pmatrix} \chi_1 \\ \frac{\sigma.\mathbf{p}}{p^0+m}\chi_1 \end{pmatrix}\begin{pmatrix} \chi_1^\dagger & \chi_1^\dagger \frac{\sigma.\mathbf{p}}{p^0+m} \end{pmatrix}\begin{pmatrix} 1_2 & 0_2 \\ 0_2 & -1_2 \end{pmatrix} \\
=& (p^0+m)\begin{pmatrix} \chi_1 \\ \frac{\sigma.\mathbf{p}}{p^0+m}\chi_1 \end{pmatrix}\begin{pmatrix} \chi_1^\dagger & -\chi_1^\dagger \frac{\sigma.\mathbf{p}}{p^0+m} \end{pmatrix} \\
=& (p^0+m)\begin{pmatrix} \chi_1\chi_1^\dagger & -\chi_1\chi_1^\dagger\frac{\sigma.\mathbf{p}}{p^0+m} \\ \frac{\sigma.\mathbf{p}}{p^0+m}\chi_1\chi_1^\dagger & -\frac{\sigma.\mathbf{p}}{p^0+m}\chi_1\chi_1^\dagger\frac{\sigma.\mathbf{p}}{p^0+m} \end{pmatrix}
\end{aligned}
$$

So

$$
\begin{aligned}
&u_1(p)\overline{u_1(p)} + u_2(p)\overline{u_2(p)} \\
=&(p^0+m)\begin{pmatrix} (\chi_1\chi_1^\dagger + \chi_2\chi_2^\dagger) & -(\chi_1\chi_1^\dagger + \chi_2\chi_2^\dagger)\frac{\sigma.\mathbf{p}}{p^0+m} \\ \frac{\sigma.\mathbf{p}}{p^0+m}(\chi_1\chi_1^\dagger + \chi_2\chi_2^\dagger) & -\frac{\sigma.\mathbf{p}}{p^0+m}(\chi_1\chi_1^\dagger + \chi_2\chi_2^\dagger)\frac{\sigma.\mathbf{p}}{p^0+m} \end{pmatrix} \\
=&(p^0+m)\begin{pmatrix} 1 & -\frac{\sigma.\mathbf{p}}{p^0+m} \\ \frac{\sigma.\mathbf{p}}{p^0+m} & -\frac{(\sigma.\mathbf{p})^2}{(p^0+m)^2} \end{pmatrix} \\
=&(p^0+m)\begin{pmatrix} 1 & -\frac{\sigma.\mathbf{p}}{p^0+m} \\ \frac{\sigma.\mathbf{p}}{p^0+m} & -\frac{(\mathbf{p})^2}{(p^0+m)^2} \end{pmatrix} \\
=&(p^0+m)\begin{pmatrix} 1 & -\frac{\sigma.\mathbf{p}}{p^0+m} \\ \frac{\sigma.\mathbf{p}}{p^0+m} & -\frac{(p^0)^2-m^2}{(p^0+m)^2} \end{pmatrix} \\
=&(p^0+m)\begin{pmatrix} 1 & -\frac{\sigma.\mathbf{p}}{p^0+m} \\ \frac{\sigma.\mathbf{p}}{p^0+m} & -\frac{p^0-m}{p^0+m} \end{pmatrix} \\
=&\begin{pmatrix} p^0+m & -\sigma.\mathbf{p} \\ \sigma.\mathbf{p} & -p^0+m \end{pmatrix} \\
=&\eta_{\mu\nu}\gamma^\mu p^\nu + m = \gamma.p + m. \\
\text{So}\quad P_+ =&\frac{1}{2m}(u_1(p)\overline{u_1(p)} + u_2(p)\overline{u_2(p)}) \\
=&\frac{1}{2m}(\gamma.p+m).
\end{aligned}
$$

(We have used $\chi_1\chi_1^\dagger + \chi_2\chi_2^\dagger = 1_2$.) So

$$
\begin{aligned}
P_+^2 =&\frac{1}{4m^2}(u_1(p)\overline{u_1(p)} + u_2(p)\overline{u_2(p)})(u_1(p)\overline{u_1(p)} + u_2(p)\overline{u_2(p)}) \\
=&\frac{1}{4m^2}[(u_1(p)\overline{u_1(p)}u_1(p)\overline{u_1(p)} + u_1(p)\overline{u_1(p)}u_2(p)\overline{u_2(p)} \\
&+ u_2(p)\overline{u_2(p)}u_1(p)\overline{u_1(p)} + u_2(p)\overline{u_2(p)}u_2(p)\overline{u_2(p)}] \\
=&\frac{1}{4m^2}(u_1(p)2m\overline{u_1(p)} + 0 + 0 + u_2(p)2m\overline{u_2(p)}) \\
=&\frac{1}{2m}(u_1(p)\overline{u_1(p)} + u_2(p)\overline{u_2(p)}) = P_+.
\end{aligned}
$$

Or

$$
\begin{aligned}
P_+^2 =&\frac{1}{4m^2}(\gamma.p+m)^2 = \frac{1}{4m^2}[(\gamma.p)^2 + 2m\gamma.p + m^2] \\
=&\frac{1}{4m^2}[p^2 + 2m\gamma.p + m^2] = \frac{1}{4m^2}[m^2 + 2m\gamma.p + m^2] \\
=&\frac{1}{4m^2}[2m^2 + 2m\gamma.p] = \frac{1}{2m}[\gamma.p + m] = P_+.
\end{aligned}
$$

Similarly,

$$\begin{aligned}
v_1(p)\overline{v_1(p)} =& v_1(p)v_1^\dagger\gamma^0 = (p^0+m)\begin{pmatrix} \frac{\sigma.\mathbf{p}}{p^0+m}\chi_1 \\ \chi_1 \end{pmatrix} \begin{pmatrix} \chi_1^\dagger \frac{\sigma.\mathbf{p}}{p^0+m} & \chi_1^\dagger \end{pmatrix} \begin{pmatrix} 1_2 & 0_2 \\ 0_2 & -1_2 \end{pmatrix} \\
=& (p^0+m)\begin{pmatrix} \frac{\sigma.\mathbf{p}}{p^0+m}\chi_1 \\ \chi_1 \end{pmatrix} \begin{pmatrix} \chi_1^\dagger \frac{\sigma.\mathbf{p}}{p^0+m} & -\chi_1^\dagger \end{pmatrix} \\
=& (p^0+m)\begin{pmatrix} \frac{\sigma.\mathbf{p}}{p^0+m}\chi_1\chi_1^\dagger \frac{\sigma.\mathbf{p}}{p^0+m} & -\frac{\sigma.\mathbf{p}}{p^0+m}\chi_1\chi_1^\dagger \\ \chi_1\chi_1^\dagger \frac{\sigma.\mathbf{p}}{p^0+m} & -\chi_1\chi_1^\dagger \end{pmatrix}
\end{aligned}$$

So

$$\begin{aligned}
& v_1(p)\overline{v_1(p)} + v_2(p)\overline{v_2(p)} \\
=& (p^0+m)\begin{pmatrix} \frac{\sigma.\mathbf{p}}{p^0+m}(\chi_1\chi_1^\dagger + \chi_2\chi_2^\dagger)\frac{\sigma.\mathbf{p}}{p^0+m} & -\frac{\sigma.\mathbf{p}}{p^0+m}(\chi_1\chi_1^\dagger + \chi_2\chi_2^\dagger) \\ (\chi_1\chi_1^\dagger + \chi_2\chi_2^\dagger)\frac{\sigma.\mathbf{p}}{p^0+m} & -(\chi_1\chi_1^\dagger + \chi_2\chi_2^\dagger) \end{pmatrix} \\
=& (p^0+m)\begin{pmatrix} \frac{(\sigma.\mathbf{p})^2}{(p^0+m)^2} & -\frac{\sigma.\mathbf{p}}{p^0+m} \\ \frac{\sigma.\mathbf{p}}{p^0+m} & -1 \end{pmatrix} \\
=& (p^0+m)\begin{pmatrix} \frac{(\mathbf{p})^2}{(p^0+m)^2} & -\frac{\sigma.\mathbf{p}}{p^0+m} \\ \frac{\sigma.\mathbf{p}}{p^0+m} & -1 \end{pmatrix} \\
=& (p^0+m)\begin{pmatrix} \frac{(p^0)^2-m^2}{(p^0+m)^2} & -\frac{\sigma.\mathbf{p}}{p^0+m} \\ \frac{\sigma.\mathbf{p}}{p^0+m} & -1 \end{pmatrix} \\
=& (p^0+m)\begin{pmatrix} \frac{p^0-m}{p^0+m} & -\frac{\sigma.\mathbf{p}}{p^0+m} \\ \frac{\sigma.\mathbf{p}}{p^0+m} & -1 \end{pmatrix} \\
=& \begin{pmatrix} p^0-m & -\sigma.\mathbf{p} \\ \sigma.\mathbf{p} & -p^0-m \end{pmatrix} \\
=& \gamma.p - m.
\end{aligned}$$

$$\begin{aligned}
\text{So} \quad P_- =& -\frac{1}{2m}(v_1(p)\overline{v_1(p)} + v_2(p)\overline{v_2(p)}) \\
=& \frac{1}{2m}(-\gamma.p + m).
\end{aligned}$$

So (this wasn't asked for in the question but we'll do it anyway)

$$\begin{aligned}
P_-^2 =& \frac{1}{4m^2}(v_1(p)\overline{v_1(p)} + v_2(p)\overline{v_2(p)})(v_1(p)\overline{v_1(p)} + v_2(p)\overline{v_2(p)}) \\
=& \frac{1}{4m^2}[v_1(p)\overline{v_1(p)}v_1(p)\overline{v_1(p)} + v_1(p)\overline{v_1(p)}v_2(p)\overline{v_2(p)} \\
& + v_2(p)\overline{v_2(p)}v_1(p)\overline{v_1(p)} + v_2(p)\overline{v_2(p)}v_2(p)\overline{v_2(p)}] \\
=& \frac{1}{4m^2}(v_1(p)(-2m)\overline{v_1(p)} + 0 + 0 + v_2(p)(-2m)\overline{v_2(p)}) \\
=& -\frac{1}{2m}(v_1(p)\overline{v_1(p)} + v_2(p)\overline{v_2(p)}) = P_-.
\end{aligned}$$

Or

$$
\begin{aligned}
P_-^2 =& \frac{1}{4m^2}(-\gamma.p + m)^2 = \frac{1}{4m^2}[(\gamma.p)^2 - 2m\gamma.p + m^2] \\
=& \frac{1}{4m^2}[p^2 - 2m\gamma.p + m^2] = \frac{1}{4m^2}[m^2 - 2m\gamma.p + m^2] \\
=& \frac{1}{4m^2}[2m^2 - 2m\gamma.p] = \frac{1}{2m}[-\gamma.p + m] = P_-.
\end{aligned}
$$