MATH425 Quantum Field Theory Solutions 5

1.

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \Rightarrow \{\gamma^\mu, \gamma_\nu\} = 2\delta^\mu{}_\nu.$$

So

$$\begin{aligned}\gamma^\mu\gamma_\lambda\gamma_\rho =& -\gamma_\lambda\gamma^\mu\gamma_\rho + 2\delta^\mu{}_\lambda\gamma_\rho \\ =& \gamma_\lambda\gamma_\rho\gamma^\mu - 2\delta^\mu{}_\rho\gamma_\lambda + 2\delta^\mu{}_\lambda\gamma_\rho \\ \Rightarrow [\gamma^\mu, \gamma_\lambda\gamma_\rho] =& 2(\delta^\mu{}_\lambda\gamma_\rho - \delta^\mu{}_\rho\gamma_\lambda).\end{aligned}$$

So

$$\begin{aligned}[\gamma^\mu, \gamma_\rho\gamma_\lambda] =& 2(\delta^\mu{}_\rho\gamma_\lambda - \delta^\mu{}_\lambda\gamma_\rho) \\ \text{and} \quad [\gamma^\mu, \gamma_\lambda\gamma_\rho - \gamma_\rho\gamma_\lambda] =& 4(\delta^\mu{}_\lambda\gamma_\rho - \delta^\mu{}_\rho\gamma_\lambda).\end{aligned}$$

Multiplying by $\frac{1}{8}$, we can write as

$$\begin{aligned}i[\gamma^\mu, -\tfrac{1}{8}i[\gamma_\lambda, \gamma_\rho]] =& \tfrac{1}{2}(\delta^\mu{}_\lambda\gamma_\rho - \delta^\mu{}_\rho\gamma_\lambda), \\ \text{i.e.} \quad i[\gamma^\mu, \Sigma_{\lambda\rho}] =& \tfrac{1}{2}(\delta^\mu{}_\lambda\gamma_\rho - \delta^\mu{}_\rho\gamma_\lambda).\end{aligned}$$

where

$$\Sigma_{\lambda\rho} = -\tfrac{1}{8}i[\gamma_\lambda, \gamma_\rho].$$

2.

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}1_4 \Rightarrow \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}1_4,$$

where 1_4 is the 4-dimensional identity matrix, usually not written explicitly. Taking the trace, and using $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, $\mathrm{tr}1_4 = 4$, we get

$$\mathrm{tr}[\gamma_\mu\gamma_\nu] = 4\eta_{\mu\nu}.$$

Now

$$\begin{aligned}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma =& -\gamma_\nu\gamma_\mu\gamma_\rho\gamma_\sigma + 2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma \\ =& \gamma_\nu\gamma_\rho\gamma_\mu\gamma_\sigma - 2\eta_{\mu\rho}\gamma_\nu\gamma_\sigma + 2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma \\ =& -\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu + 2\eta_{\mu\sigma}\gamma_\nu\gamma_\rho - 2\eta_{\mu\rho}\gamma_\nu\gamma_\sigma + 2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma \\ \Rightarrow \gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma + \gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu =& 2\eta_{\mu\sigma}\gamma_\nu\gamma_\rho - 2\eta_{\mu\rho}\gamma_\nu\gamma_\sigma + 2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma.\end{aligned}$$

Taking the trace and using

$$\mathrm{tr}[\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu] = \mathrm{tr}[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma],$$

together with $\mathrm{tr}[\gamma_\mu\gamma_\nu = 4\eta_{\mu\nu}$, we find

$$\mathrm{tr}[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma] = 4[\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}]$$

3.

$$\begin{aligned}\gamma_\mu \not{a}_1 \gamma^\mu =& a_1^\nu\gamma_\mu\gamma_\nu\gamma^\mu = a_1^\nu(-\gamma_\nu\gamma_\mu + 2\eta_{\mu\nu})\gamma^\mu \\ =& a_1^\nu(-\gamma_\nu\gamma_\mu\gamma^\mu + 2\gamma_\nu).\end{aligned}$$

Now

$$\begin{aligned}\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu =& 2\eta_{\mu\nu},\\ \Rightarrow \eta^{\mu\nu}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) =& 2\eta^{\mu\nu}\eta_{\mu\nu}\\ \Rightarrow \gamma_\mu\gamma^\mu + \gamma_\nu\gamma^\nu =& 2\delta^\mu{}_\mu = 8 \Rightarrow 2\gamma_\mu\gamma^\mu = 8 \Rightarrow \gamma_\mu\gamma^\mu = 4.\end{aligned}$$

So we have

$$\gamma_\mu \not{a}_1 \gamma^\mu = (-4+2)a_1^\nu\gamma_\nu = -2(a_1.\gamma).$$

Also

$$\begin{aligned}\gamma_\mu \not{a}_1 \not{a}_2 \gamma^\mu =& a_1^\nu a_2^\rho \gamma_\mu\gamma_\nu\gamma_\rho\gamma^\mu\\ =& a_1^\nu a_2^\rho(-\gamma_\nu\gamma_\mu + 2\eta_{\mu\nu})\gamma_\rho\gamma^\mu\\ =& a_1^\nu a_2^\rho(-\gamma_\nu\gamma_\mu\gamma_\rho\gamma^\mu + 2\gamma_\rho\gamma_\nu.\end{aligned}$$

Now we know that $\gamma_\mu\gamma_\rho\gamma^\mu = -2\gamma_\rho$, so we have

$$\gamma_\mu \not{a}_1 \not{a}_2 \gamma^\mu = 2a_1^\nu a_2^\rho(\gamma_\nu\gamma_\rho + \gamma_\rho\gamma_\nu) = 4a_1^\nu a_2^\rho \eta_{\nu\rho} = 4a_1.a_2.$$

4. Under a Lorentz transformation we have

$$\begin{aligned}j'^\mu(x') =& \overline{\psi'(x')}\gamma^5\gamma^\mu\psi'(x') = \overline{\psi(x)}S(L)^{-1}\gamma^5\gamma^\mu S(L)\psi(x)\\ =& \overline{\psi(x)}S(L)^{-1}\gamma^5 S(L)S(L)^{-1}\gamma^\mu S(L)\psi(x) = L^\mu{}_\nu\overline{\psi(x)}\gamma^5\gamma^\nu\psi(x) = L^\mu{}_\nu j^\nu(x),\end{aligned}$$

using $S(L)^{-1}\gamma^5 S(L) = \gamma^5$, $S(L)^{-1}\gamma^\mu S(L) = L^\mu{}_\nu\gamma^\nu$. Under a parity transformation we have

$$\begin{aligned}j'^\mu(x') =& \overline{\psi'(x')}\gamma^5\gamma^\mu\psi'(x') = \overline{\psi(x)}S(P)^{-1}\gamma^5\gamma^\mu S(P)\psi(x)\\ =& \overline{\psi(x)}S(P)^{-1}\gamma^5 S(P)S(P)^{-1}\gamma^\mu S(P)\psi(x)\\ =& -P^\mu{}_\nu\overline{\psi(x)}\gamma^5\gamma^\nu\psi(x) = -P^\mu{}_\nu j^\nu(x),\end{aligned}$$

using $S(P)^{-1}\gamma^5 S(P) = -\gamma^5$, $S(P)^{-1}\gamma^\mu S(P) = P^\mu{}_\nu\gamma^\nu$.