# MATH425 Quantum Field Theory Solutions 3

1.

$$\begin{aligned} L &= \begin{pmatrix} \gamma & -\frac{\gamma v}{c} \\ -\frac{\gamma v}{c} & \gamma \end{pmatrix} \\ \Rightarrow L^T &= L. \\ \text{So} \quad L^T \eta L &= \begin{pmatrix} \gamma & -\frac{\gamma v}{c} \\ -\frac{\gamma v}{c} & \gamma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \gamma & -\frac{\gamma v}{c} \\ -\frac{\gamma v}{c} & \gamma \end{pmatrix} \\ &= \begin{pmatrix} \gamma & -\frac{\gamma v}{c} \\ -\frac{\gamma v}{c} & \gamma \end{pmatrix} \begin{pmatrix} \gamma & -\frac{\gamma v}{c} \\ \frac{\gamma v}{c} & -\gamma \end{pmatrix} \\ &= \begin{pmatrix} \gamma^2 - \left(\frac{\gamma v}{c}\right)^2 & -\gamma\frac{\gamma v}{c} + \gamma\frac{\gamma v}{c} \\ -\gamma\frac{\gamma v}{c} + \gamma\frac{\gamma v}{c} & \left(\frac{\gamma v}{c}\right)^2 - \gamma^2 \end{pmatrix} \\ &= \begin{pmatrix} \gamma^2\left(1 - \frac{v^2}{c^2}\right) & 0 \\ 0 & -\gamma^2\left(1 - \frac{v^2}{c^2}\right) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \eta, \end{aligned}$$

since

$$\gamma^2 = \frac{1}{1 - \frac{v^2}{c^2}}.$$

2(i). Taking the inverse of (1), we have

$$(L^T \eta L)^{-1} = \eta^{-1} \Rightarrow L^{-1}\eta^{-1}(L^T)^{-1} = \eta^{-1}$$

Multiplying on the left by $L$ and on the right by $L^T$, we find

$$\begin{aligned} LL^{-1}\eta^{-1}(L^T)^{-1}L^T &= L\eta^{-1}L^T \\ \eta^{-1} &= L\eta^{-1}L^T, \end{aligned}$$

as required.

(ii) Putting $\mu = \nu = 0$ in (3), we have

$$\begin{aligned} \eta_{\alpha\beta}L^\alpha{}_0 L^\beta{}_0 = 1, &\quad \eta^{\alpha\beta}L^0{}_\alpha L^0{}_\beta = 1 \\ \text{i.e. } \left(L^0{}_0\right)^2 - \left(L^1{}_0\right)^2 - \left(L^2{}_0\right)^2 - \left(L^3{}_0\right)^2 &= 1, \\ \left(\bar{L}^0{}_0\right)^2 - \left(\bar{L}^0{}_1\right)^2 - \left(\bar{L}^0{}_2\right)^2 - \left(\bar{L}^0{}_3\right)^2 &= 1 \\ \Rightarrow \left(L^0{}_0\right)^2 - |\mathbf{l}|^2 = 1, &\quad \left(\bar{L}^0{}_0\right)^2 - |\bar{\mathbf{l}}|^2 = 1 \\ \Rightarrow |\mathbf{l}| = \sqrt{\left(L^0{}_0\right)^2 - 1}, &\quad |\bar{\mathbf{l}}| = \sqrt{\left(\bar{L}^0{}_0\right)^2 - 1}. \end{aligned}$$

(iii)

$$\begin{aligned} (\bar{L}L)^0{}_0 &= \bar{L}^0{}_\alpha L^\alpha{}_0 \\ &= \bar{L}^0{}_0 L^0{}_0 + \bar{L}^0{}_1 L^1{}_0 + \bar{L}^0{}_2 L^2{}_0 + \bar{L}^0{}_3 L^3{}_0 \\ &= \bar{L}^0{}_0 L^0{}_0 + \bar{\mathbf{l}}.\mathbf{l} \end{aligned}$$

(iv)

$$|\bar{\mathbf{l}}.\mathbf{l}| \le |\mathbf{l}||\bar{\mathbf{l}}| \Rightarrow -|\mathbf{l}||\bar{\mathbf{l}}| \le \bar{\mathbf{l}}.\mathbf{l} \le |\mathbf{l}||\bar{\mathbf{l}}|$$

$$\Rightarrow \quad \text{(using (iii))} \quad (\bar{L}L)^0{}_0 - \bar{L}^0{}_0 L^0{}_0 \ge -|\mathbf{l}||\bar{\mathbf{l}}|$$

$$\Rightarrow (\bar{L}L)^0{}_0 \ge \bar{L}^0{}_0 L^0{}_0 - |\mathbf{l}||\bar{\mathbf{l}}|$$

$$\text{i.e. (using (ii))} \quad (\bar{L}L)^0{}_0 \ge \bar{L}^0{}_0 L^0{}_0 - \sqrt{\left(L^0{}_0\right)^2 - 1}\sqrt{\left(\bar{L}^0{}_0\right)^2 - 1}.$$

(v)

$$(x-y)^2 \ge 0 \Rightarrow x^2 - 2xy + y^2 \ge 0 \Rightarrow -2xy \ge -x^2 - y^2$$

$$\Rightarrow x^2y^2 - 2xy + 1 \ge x^2y^2 - x^2 - y^2 + 1$$

$$\Rightarrow x^2y^2 - 2xy + 1 \ge (x^2-1)(y^2-1) \Rightarrow (xy-1)^2 \ge (x^2-1)(y^2-1).$$

$$\Rightarrow \text{either} \quad xy - 1 \ge \sqrt{x^2-1}\sqrt{y^2-1} \quad \text{or} \quad xy - 1 \le -\sqrt{x^2-1}\sqrt{y^2-1}.$$

If $x, y \ge 1$ then $xy - 1$ is positive, and we must have the first inequality, implying

$$xy - \sqrt{x^2-1}\sqrt{y^2-1} \ge 1.$$

Writing $L^0{}_0 = x$, $\bar{L}^0{}_0 = y$, we have from (iv)

$$(\bar{L}L)^0{}_0 \ge \bar{L}^0{}_0 L^0{}_0 - \sqrt{\left(L^0{}_0\right)^2 - 1}\sqrt{\left(\bar{L}^0{}_0\right)^2 - 1} \ge 1.$$

(vi) Obviously if $\det \bar{L} = \det L = 1$, then

$$\det(\bar{L}L) = \det \bar{L} \det L = 1.$$

(vii) We have now shown that if $L \in \mathcal{L}_+^\uparrow$, $L \in \mathcal{L}_+^\uparrow$, then $\bar{L}L \in \mathcal{L}_+^\uparrow$. It is clear that $1 \in \mathcal{L}_+^\uparrow$.

(viii) We can write (1) as

$$(\eta^{-1}L^T\eta)L = \eta^{-1}\eta = 1,$$

which shows that $L^{-1} = \eta^{-1}L^T\eta$, i.e. $(L^{-1})^\mu{}_\nu = \eta^{\mu\alpha}(L^T)_\alpha{}^\beta\eta_{\beta\nu} = \eta^{\mu\alpha}L^\beta{}_\alpha\eta_{\beta\nu}$. So $(L^{-1})^0{}_0 = \eta^{00}L^0{}_0\eta_{00} = L^0{}_0$. Moreover,

$$\det L^{-1} = \det \eta^{-1} \det L^T \det \eta = (-1)\det L(-1) = \det L = 1.$$

So $L^{-1} \in \mathcal{L}_+^\uparrow$. The remaining group property is associativity, $(L_1L_2)L_3 = L_1(L_2L_3)$, which is true for all matrices. So $\mathcal{L}_+^\uparrow$ *is* a group.