MATH425 - Quantum Field Theory

Solutions Set Work Sheet 2

1. In the interaction picture the Hamiltonian is given by

$$H = H_0 + H_I$$

where the free Hamiltonian $H_0^I = H_0^S$ is the same as in the Schrödinger picture. A state in the Interaction picture is related to a state in the SChrödinger picture by

$$|A,t\rangle_I = U_0^\dagger |A,t\rangle_S$$

where,

$$U_0 \equiv U_0(t,t_0) = \mathrm{e}^{-i\frac{H_0(t-t_0)}{\hbar}},$$

and an operator in the interaction picture is related to an operator in the Schrödinger picture by

$$O^I(t) = U_0^\dagger O^S U_0.$$

(i) We have to show that an operator in the interaction picture obeys

$$i\hbar \frac{dO^I(t)}{dt} = \left[O^I(t), H_0\right]$$

Start from

$$O^I(t) = U_0^\dagger O^S U_0$$

differentiate both sides with respect to t

$$i\hbar\frac{dO^I(t)}{dt} = i\hbar(\frac{dU_0^\dagger}{dt}U_0U_0^\dagger O^S U_0 + U_0^\dagger O^S U_0 U_0^\dagger \frac{dU_0}{dt}) = i\hbar(\frac{dU_0^\dagger}{dt}U_0 O^I + O^I U_0^\dagger \frac{dU_0}{dt})$$

remembering that O^S is time independent. We have the relations

$$i\hbar\frac{dU_0}{dt} = H_0 U_0 \Rightarrow i\hbar\frac{dU_0}{dt}U_0^\dagger = H_0$$

and

$$-i\hbar\frac{dU_0^\dagger}{dt} = U_0^\dagger H_0 \Rightarrow -i\hbar U_0\frac{dU_0^\dagger}{dt} = H_0$$

we get

$$(-U_0^\dagger H_0 U_0 O^I + O^I U_0^\dagger H_0 U_0) = -H_0 O^I + O^I H_0 = [O^I, H_0]$$

(ii) We have to show that a state in the interaction picture obeys

$$i\hbar\frac{d}{dt}\frac{|A,t\rangle_I}{} = H_I^I |A,t\rangle_I\ .$$

$$i\hbar\frac{d}{dt}\frac{|A,t\rangle_S}{} = H^S|A,t\rangle_S \ .$$

$$|A,t\rangle_S = U_0^\dagger|A,t\rangle_I$$

$$i\hbar\frac{d}{dt}(U_0|A,t\rangle_I) = (H_0^S + H_I^S)U_0|A,t\rangle_I$$

$$U_0^\dagger i\hbar\frac{d}{dt}(|A,t\rangle_I) + i\hbar(\frac{dU_0}{dt})|A,t\rangle_I = H_0^S U_0|A,t\rangle_I + H_I^S U_0|A,t\rangle_I$$

$$i\hbar\frac{d}{dt}(|A,t\rangle_I) + U_0^\dagger i\hbar(\frac{dU_0}{dt})|A,t\rangle_I = U_0^\dagger H_0^S U_0|A,t\rangle_I + U_0^\dagger H_I^S U_0|A,t\rangle_I$$

$$i\hbar\frac{d}{dt}(|A,t\rangle_I) + U_0^\dagger H_0^S U_0|A,t\rangle_I = U_0^\dagger H_0^S U_0|A,t\rangle_I + U_0^\dagger H_I^S U_0|A,t\rangle_I$$

$$i\hbar\frac{d}{dt}(|A,t\rangle_I) = H_I^I|A,t\rangle_I$$

2.

$$\begin{aligned}
L &= \begin{pmatrix} \gamma & -\frac{\gamma v}{c} \\ -\frac{\gamma v}{c} & \gamma \end{pmatrix} \\
\Rightarrow L^T &= L. \\
\text{So} \quad L^T\eta L &= \begin{pmatrix} \gamma & -\frac{\gamma v}{c} \\ -\frac{\gamma v}{c} & \gamma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \gamma & -\frac{\gamma v}{c} \\ -\frac{\gamma v}{c} & \gamma \end{pmatrix} \\
&= \begin{pmatrix} \gamma & -\frac{\gamma v}{c} \\ -\frac{\gamma v}{c} & \gamma \end{pmatrix} \begin{pmatrix} \gamma & -\frac{\gamma v}{c} \\ \frac{\gamma v}{c} & -\gamma \end{pmatrix} \\
&= \begin{pmatrix} \gamma^2 - \left(\frac{\gamma v}{c}\right)^2 & -\gamma\frac{\gamma v}{c} + \gamma\frac{\gamma v}{c} \\ -\gamma\frac{\gamma v}{c} + \gamma\frac{\gamma v}{c} & \left(\frac{\gamma v}{c}\right)^2 - \gamma^2 \end{pmatrix} \\
&= \begin{pmatrix} \gamma^2\left(1 - \frac{v^2}{c^2}\right) & 0 \\ 0 & -\gamma^2\left(1 - \frac{v^2}{c^2}\right) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \eta,
\end{aligned}$$

since

$$\gamma^2 = \frac{1}{1 - \frac{v^2}{c^2}}.$$