QUANTUM FIELD THEORY

1. Relativistic Quantum Mechanics

Basics

After quantum mechanics was first formulated, the most urgent problem was to devise a relativistic generalisation of Schrödinger's equation. The Klein-Gordon equation was the first attempt. First let us review special relativity in order to establish our notation. We write the position 4-vector

$$x = (x^0, x^1, x^2, x^3) = (ct, \mathbf{x}), \tag{1.1}$$

and we define the Minkowski metric

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.2}$$

The scalar product of two 4-vectors is defined by

$$x.y = \eta_{\mu\nu} x^\mu y^\nu, \tag{1.3}$$

and the Lorentz group consists of linear transformations preserving $x.y$:

$$\begin{aligned} x \to x' \quad x'^\alpha &= L^\alpha{}_\nu x^\nu \\ y \to y' \quad y'^\beta &= L^\beta{}_\nu y^\nu \\ x'.y' = x.y \Rightarrow \eta_{\alpha\beta} x'^\alpha y'^\beta = \eta_{\alpha\beta} L^\alpha{}_\mu L^\beta{}_\nu x^\mu y^\nu &= \eta_{\mu\nu} x^\mu y^\nu \quad \text{for all} \quad x^\mu, y^\nu \\ \Rightarrow \eta_{\alpha\beta} L^\alpha{}_\mu L^\beta{}_\nu &= \eta_{\mu\nu}. \end{aligned} \tag{1.4}$$

We define

$$\begin{aligned} \mathcal{L} &= \{L : \eta_{\alpha\beta} L^\alpha{}_\mu L^\beta{}_\nu = \eta_{\mu\nu}\} \\ \mathcal{L}^\uparrow &= \{L \in \mathcal{L} : L^0{}_0 \geq 1\} \\ \mathcal{L}^\downarrow &= \{L \in \mathcal{L} : L^0{}_0 \leq -1\} \\ \mathcal{L}^+ &= \{L \in \mathcal{L} : \det L = +1\} \\ \mathcal{L}^- &= \{L \in \mathcal{L} : \det L = -1\}. \end{aligned} \tag{1.5}$$

The Lorentz group splits up into disconnected components $\mathcal{L}_+^\uparrow$, $\mathcal{L}_-^\uparrow$, $\mathcal{L}_+^\downarrow$, $\mathcal{L}_-^\downarrow$. $\mathcal{L}_+^\uparrow$ is called the "proper orthochronous Lorentz group". Any element of $\mathcal{L}_+^\uparrow$ may be reached from $L = 1$ by continuously varying the elements of $L \in \mathcal{L}$. Special relativity requires that theories should be invariant under transformations

$$x^\mu \to x'^\mu = L^\mu{}_\nu x^\nu, \quad L \in \mathcal{L}_+^\uparrow. \tag{1.6}$$

We shall call $\mathcal{L}_+^\uparrow$ the Lorentz group and $\mathcal{L}$ the full Lorentz group. Since we can build up any $L \in \mathcal{L}_+^\uparrow$ by a sequence of infinitesimal transformations, it is sufficient to check invariance under these:

$$L^\mu{}_\nu = \delta^\mu{}_\nu + \epsilon^\mu{}_\nu, \tag{1.7}$$

where $\epsilon^\mu{}_\nu$ is small. Then

$$\eta_{\alpha\beta} L^\alpha{}_\mu L^\beta{}_\nu = \eta_{\mu\nu} \Rightarrow \eta_{\mu\nu} + \epsilon_{\mu\nu} + \epsilon_{\nu\mu} = \eta_{\mu\nu}. \tag{1.8}$$

So the condition for $L = 1 + \epsilon \in \mathcal{L}$ is

$$\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}, \tag{1.9}$$

i.e. ϵ is antisymmetric.

The Klein-Gordon Equation

In non-relativistic quantum mechanics, $\mathbf{p} \to -i\hbar\nabla$, $E \to i\hbar\frac{\partial}{\partial t}$. The relation

$$\frac{\mathbf{p}^2}{2m} = E = H \Rightarrow -\frac{\hbar^2}{2m}\nabla^2\psi = i\hbar\frac{\partial\psi}{\partial t}, \tag{1.10}$$

the Schrödinger equation. In special relativity, the 4-vector p^μ is given by $p^\mu = (\frac{E}{c}, \mathbf{p})$. But $\partial^\mu = (\frac{1}{c}\frac{\partial}{\partial t}, -\nabla)$. So we can identify $p^\mu \to i\hbar\partial^\mu$ which is Lorentz covariant. In special relativity

$$\begin{aligned} E =& \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}, \quad \mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}}, \\ \text{so} \quad p^2 = \frac{E^2}{c^2} - \mathbf{p}^2 =& m^2c^2 \end{aligned} \tag{1.11}$$

Following the example of non-relativistic quantum mechanics, it is tempting to use $p^\mu \to i\hbar\partial^\mu$ to translate this relation into the wave equation

$$\begin{aligned} -\hbar^2\partial^2\phi =& m^2c^2\phi \\ \text{or} \quad \left(\partial^2 + \frac{m^2c^2}{\hbar^2}\right)\phi =& 0. \end{aligned} \tag{1.12}$$

This is the **Klein-Gordon equation**. We shall now go on to discuss the problems which arise in trying to interpret this as a relativistic Schrödinger equation.

Solutions

A basic set of solutions is formed by the "plane wave" solutions $\phi = Ae^{-ik.x}$, k constant. We have

$$\begin{aligned}\partial^\mu\phi = \frac{\partial}{\partial x_\mu}\phi =& - ik^\mu Ae^{-ik.x} = -ik^\mu\phi \\ \partial^2\phi =& - k^2\phi.\end{aligned} \tag{1.13}$$

For a solution we need $k^2 = \frac{m^2c^2}{\hbar^2}$. We have

$$p^\mu\phi = i\hbar\partial^\mu\phi = \hbar k^\mu\phi. \tag{1.14}$$

ϕ describes an e-state of momentum with e-value $\hbar k^\mu$ and the condition $k^2 = \frac{m^2c^2}{\hbar^2}$ is the same as the condition $p^2 = m^2c^2$, the "mass-shell condition". Since we have

$$p^2 = \frac{E^2}{c^2} - \mathbf{p}^2 = m^2c^2 \Rightarrow \frac{E}{c} = \pm\sqrt{m^2c^2 + \mathbf{p}^2}, \tag{1.15}$$

we see that some of the solutions to this equation correspond to negative energy states.

From now on we set $\hbar = c = 1$ and restore them where necessary (by dimensional arguments).

We write the general solution to the K-G equation in the form

$$\phi(x) = \int d^4p g(p)e^{-ip.x}.$$

Then

$$\begin{aligned}(\partial^2 + m^2)\phi = 0 \Rightarrow \int d^4p(-p^2 + m^2)g(p)e^{-ip.x} =& 0 \\ \Rightarrow (p^2 - m^2)g(p) =& 0 \\ \Rightarrow g(p) =& \frac{1}{(2\pi)^3}f(p)\delta(p^2 - m^2) \\ \text{So} \quad \phi(x) =& \frac{1}{(2\pi)^3}\int d^4pe^{-ip.x}\delta(p^2 - m^2)f(p) \\ =& \frac{1}{(2\pi)^3}\int d^3\mathbf{p}\frac{1}{2p^0}\left[f_+(\mathbf{p})e^{-ip.x} + f_-(\mathbf{p})e^{ip.x}\right]\end{aligned} \tag{1.16}$$

where $p^0 = \sqrt{\mathbf{p}^2 + m^2}$, and

$$\begin{aligned}f_+(\mathbf{p}) =& f(+\sqrt{\mathbf{p}^2 + m^2}, \mathbf{p}), \\ f_-(\mathbf{p}) =& f(-\sqrt{\mathbf{p}^2 + m^2}, -\mathbf{p}).\end{aligned} \tag{1.17}$$

We have used here some rules for δ-functions:

$$\begin{aligned}\int f(x)\delta(x-a)dx =& f(a)\\ \int f(x)\delta(F(x))dx =& \int f(F^{-1}(y))\delta(y)\frac{dy}{F'(x)}\\ =& \frac{f(a)}{F'(a)},\end{aligned} \tag{1.18}$$

where a is a root of F, i.e. $F(a) = 0$.

If there is more than one root a_i, we get

$$\int f(x)\delta(F(x))dx = \sum_i \frac{f(a_i)}{|F'(a_i)|}. \tag{1.19}$$

The f_+ term corresponds to +ve energy states, while the f_- term corresponds to −ve energy states (because $i\frac{\partial}{\partial t}e^{ip.x} = -p^0 e^{ip.x}$).

If we don't want −ve energy solutions we could try just to throw away the −ve energy ones. However, in the presence of interactions it is impossible to prevent transitions to −ve energy states.

Probabilities

Non-relativistically the probability density is given by $|\psi(x)|^2 = \rho(x)$, which has the important property that it is conserved. I.e. if ψ is normalised such that $\int \rho(\mathbf{x},t)d^3x = 1$ at $t = 0$, it's true for all t. We see this as follows: We establish a conserved current

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \text{div}\mathbf{j} =& 0,\\ \text{where} \quad \mathbf{j} =& -\frac{i\hbar}{2m}\left[\psi^*\nabla\psi - \psi\nabla\psi^*\right]\end{aligned} \tag{1.20}$$

Then (1.20) follows from the Schrödinger equation provided the potential V is real. Integrating (1.20), we have

$$\begin{aligned}\left[\int \rho(\mathbf{x},t)d^3\mathbf{x}\right]_{t_1}^{t_2} &= \int_{t_1}^{t_2} \frac{\partial \rho}{\partial t} d^3\mathbf{x}dt = -\int_{t_1}^{t_2} \text{div}\mathbf{j}d^4x \\ &= -\int_{t_1}^{t_2} \left\{\int_{\partial V} \mathbf{j}.d\mathbf{S}\right\} dt \to 0 \quad \text{as} \quad V \to \infty,\end{aligned} \tag{1.21}$$

provided $\mathbf{j} \to 0$ sufficiently fast as $|\mathbf{x}| \to \infty$.

Defining $j^\mu = (j^0, \mathbf{j})$ where $j^0 = c\rho$, we can write (1.20) as $\frac{\partial j^\mu}{\partial x^\mu} = 0 = \partial_\mu j^\mu$. This equation is relativistically invariant, i.e. if it holds in one frame it holds in any other, provided that j^μ transforms as a 4-vector, i.e. like x^μ.

Relativistically we need to show that the statement

$$\int j^0(\mathbf{x}) d^3\mathbf{x} = 1 \tag{1.22}$$

is frame-independent. We have $j'^\mu(x) = L^\mu{}_\nu j^\nu(x)$, $x'^\mu = L^\mu{}_\nu x^\nu$, and we need to show that (1.22)$\Rightarrow \int j'^0 d^3\mathbf{x}' = 1$. This also follows from $\partial_\mu j^\mu = 0$. To see this we apply the divergence theorem in 4D to the region between $x^0 = 0$ and $x'^0 = 0$.

$$0 = \int_{\text{region}} \partial_\mu j^\mu = \int j^\mu n_\mu dS \tag{1.23}$$

where n_μ is the unit normal to the 3-surface. The normal to $x^0 = 0$ is $(1,0,0,0)$, and the normal to $x'^0 = L^0{}_\nu x^\nu = 0$ is $n_\nu = \frac{\partial x'^0}{\partial x^\nu} = L^0{}_\nu$. Then from (1.23),

$$\begin{aligned}\int_{x'^0=0} L^0{}_\nu j^\nu d^3\mathbf{x}' - \int_{x^0=0} j^0 d^3\mathbf{x} &= 0, \\ \text{i.e.} \quad \int j'^0 d^3\mathbf{x}' &= \int j^0 d^3\mathbf{x}.\end{aligned} \tag{1.24}$$

We need to find j such that $\partial_\mu j^\mu = 0$. We have

$$\begin{aligned}
(\partial^2 + m^2)\phi =&0 \\
\Rightarrow \phi^*(\partial^2 + m^2)\phi =&0 \\
\text{and} \quad \phi(\partial^2 + m^2)\phi^* =&0 \\
\text{Subtracting:} \quad \phi^*\partial^2\phi - \phi\partial^2\phi^* =&0 \\
\Rightarrow \partial_\mu j^\mu = 0 \quad \text{where} \quad j^\mu =&i(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*) \equiv i\phi^* \overleftrightarrow{\partial^\mu} \phi.
\end{aligned} \tag{1.25}$$

With this definition of probability density,

$$\rho(x) = i\phi^* \overleftrightarrow{\partial_0} \phi. \tag{1.26}$$

Non-relativistically,

$$\begin{aligned}
\rho(x) =&|\psi(x)|^2 \\
< \psi_1, \psi_2 >=& \int \psi_1^*(x)\psi_2(x)dx.
\end{aligned}$$

$\int \rho d^3\mathbf{x} = 1$ is the same as saying $||\psi||^2 =< \psi, \psi >= 1$. So now the natural thing is to define

$$\begin{aligned}
< \phi_1, \phi_2 >=&i \int \phi_1^* \overleftrightarrow{\partial_0} \phi_2 d^3\mathbf{x}, \\
||\phi||^2 = i \int \phi^* \overleftrightarrow{\partial_0} \phi d^3\mathbf{x}.
\end{aligned}$$

Unfortunately, ρ and $||\phi||^2$ are not obviously positive. In fact, $||\phi||^2$ *is* positive if we restrict attention to +ve energy solutions.

$$\begin{aligned}
\phi(x) =&\frac{1}{(2\pi)^3} \int d^3\mathbf{p} \frac{1}{2p^0} f_+(\mathbf{p}) e^{-ip.x} \\
||\phi||^2 =&i \int \phi^* \overleftrightarrow{\partial^0} \phi(x) d^3\mathbf{x} \\
=&\frac{i}{(2\pi)^6} \int d^3\mathbf{x} \left[\int d^3\mathbf{p}' \frac{1}{2p'^0} f_+^*(\mathbf{p}') e^{ip'.x} \right] \overleftrightarrow{\partial_0} \left[\int d^3\mathbf{p} \frac{1}{2p^0} f_+(\mathbf{p}) e^{-ip.x} \right] \\
=&\frac{1}{(2\pi)^6} \int d^3\mathbf{p} \frac{1}{2p^0} \int d^3\mathbf{p}' \frac{1}{2p'^0} (p'^0 + p^0) f_+(\mathbf{p}) f_+^*(\mathbf{p}') \int d^3\mathbf{x} e^{i(\mathbf{p}'-\mathbf{p}).\mathbf{x}} \quad (x_0 \equiv t = 0) \\
=&\frac{1}{(2\pi)^3} \int d^3\mathbf{p} \frac{1}{2p^0} \int d^3\mathbf{p}' \frac{1}{2p'^0} (p'^0 + p^0) f_+(\mathbf{p}) f_+^*(\mathbf{p}') \delta(\mathbf{p} - \mathbf{p}') \\
=&\frac{1}{(2\pi)^3} \int d^3\mathbf{p} \frac{1}{2p^0} |f_+(\mathbf{p})|^2 > 0.
\end{aligned} \tag{1.27}$$

If

$$\phi = \frac{1}{(2\pi)^3} \int d^3\mathbf{p} \frac{1}{2p^0} \left[f_+(\mathbf{p}) e^{-ip.x} + f_-(\mathbf{p}) e^{ip.x} \right]$$

then

$$||\phi||^2 = \frac{1}{(2\pi)^3} \int d^3\mathbf{p} \frac{1}{2p^0} \left[|f_+(\mathbf{p})|^2 - |f_-(\mathbf{p})|^2 \right].$$

Effect of Lorentz transformations

Consider translating a state (of momentum p):

$$\phi_p(x) = e^{-ip.x}.$$

If we translate by a displacement a then wave function of new state is

$$\phi_p(x-a) = e^{-ip.(x-a)}.$$

Similarly considering a Lorentz transformation L, the new state has momentum Lp, and

$$\begin{aligned} \phi_{Lp}(x) =& e^{-i(Lp).x} \\ =& e^{-ip.(L^{-1}x)} \\ =& \phi_p(L^{-1}x). \end{aligned}$$

Construction of the Dirac equation

Dirac believed that equations of motion in quantum mechanics should be linear in $\frac{\partial}{\partial t}$. On the other hand, relativistically $p^2 = m^2$ and $p^\mu \to i\partial^\mu$ seems $\Rightarrow$ KG equation. Dirac's approach was to factor the KG equation, writing

$$-(\partial^2 + m^2) = (i\gamma^\nu \partial_\nu + m)(i\gamma^\mu \partial_\mu - m)$$

The γ^μ are not ordinary numbers; they are matrices describing some sort of internal degrees of freedom. Then the KG equation can be replaced by

$$(i\gamma^\mu \partial_\mu - m)\psi = 0.$$

to be the KG equation. Have

$$\begin{aligned} (-\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu - m^2)\psi =& 0. \\ \gamma^\nu \gamma^\mu \partial_\nu \partial_\mu =& \tfrac{1}{2}\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu + \tfrac{1}{2}\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu \\ =& \tfrac{1}{2}\left\{\gamma^\nu, \gamma^\mu\right\} \partial_\nu \partial_\mu \\ \text{where} \quad \{\gamma^\nu, \gamma^\mu\} =& \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu, \end{aligned}$$

the anticommutator. We need

$$\tfrac{1}{2}\{\gamma^\mu,\gamma^\nu\}\,\partial_\mu\partial_\nu\psi \equiv \partial^2\psi,$$

$$\text{i.e.}\quad \{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}.$$

$$\text{or}\quad \gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu \quad \text{for} \quad \mu\neq\nu,$$

$$\left(\gamma^0\right)^2 = 1, \quad \left(\gamma^1\right)^2 = \left(\gamma^2\right)^2 = \left(\gamma^3\right)^2 = -1.$$

The Dirac equation can be written as

$$i\gamma^0\frac{\partial\psi}{\partial t} + i\gamma.\nabla\psi\psi - m\psi = 0$$

$$\Rightarrow i\frac{\partial\psi}{\partial t} = -i\gamma^0\gamma.\nabla\psi + m\gamma^0\psi$$

$$= -i\alpha.\nabla\psi + \beta\psi$$

where

$$\alpha = \gamma^0\gamma, \quad \beta = \gamma^0 m.$$

The Dirac equation is of the form

$$i\frac{\partial\psi}{\partial t} = H\psi$$

where

$$H = -i\alpha.\nabla + \beta = \alpha.\mathbf{p} + \beta.$$

Then

$$H = H^\dagger \Leftrightarrow \alpha_i = \alpha_i^\dagger, \quad \beta = \beta^\dagger$$

$$\Rightarrow \gamma^{0\dagger} = \gamma^0, \quad \gamma^0\gamma^i = \left(\gamma^0\gamma^i\right)^\dagger = \gamma^{i\dagger}\gamma^{0\dagger} = \gamma^{i\dagger}\gamma^0$$

$$\text{i.e.}\quad \gamma^0\gamma^i\gamma^0 = \gamma^{i\dagger}, \quad i = 1,2,3.$$

This can be summarised by

$$\gamma^0\gamma^\mu\gamma^0 = \gamma^{\mu\dagger}.$$

This, together with

$$\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}$$

are the defining properties of the Dirac γ matrices. Given that spacetime is 4-dimensional, the γs need to be at least 4×4 matrices. A possible representation for the γ is

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad i = 1,2,3.$$

The σ^i are the Pauli matrices which satisfy

$$\sigma^i \sigma^j = \delta_{ij} + i\epsilon_{ijk}\sigma^k.$$

Can check that defining properties of Dirac γ matrices are satisfied.

The equation

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

is called the Dirac equation.

Equivalence of different representations

Suppose we have two different sets of 4×4 matrices satisfying the above defining conditions, so we have

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \quad \gamma^0\gamma^\mu\gamma^0 = \gamma^{\mu\dagger}$$
$$\{\gamma'^\mu, \gamma'^\nu\} = 2\eta^{\mu\nu}, \quad \gamma'^0\gamma'^\mu\gamma'^0 = \gamma'^{\mu\dagger}.$$

For the Dirac equation to yield unambiguous results we need to show that the two representations lead to the same physical consequences. We need to show that there is a unitary transformation $\psi' = U\psi$ such that if

$$\begin{aligned} (i\gamma^\mu \partial_\mu - m)\psi &= 0 \\ \text{then} \quad (i\gamma'^\mu \partial_\mu - m)\psi' &= 0. \end{aligned}$$

This follows from the theorem

Theorem

If

$$\{\gamma^\mu, \gamma^\nu\} = \{\gamma'^\mu, \gamma'^\nu\} = 2\eta^{\mu\nu}$$

where γ^μ, γ'^μ are 4×4 matrices, then there exists a matrix M such that

$$\gamma'^\mu = M\gamma^\mu M^{-1}.$$

Moreover, M is uniquely determined up to a factor. In particular, if $\gamma^\mu = M\gamma^\mu M^{-1}$, then $M = \lambda I_4$.

If in addition

$$\gamma^0\gamma^\mu\gamma^0 = \gamma^{\mu\dagger}, \quad \gamma'^0\gamma'^\mu\gamma'^0 = \gamma'^{\mu\dagger},$$

then M can be taken to be unitary, and we have $\gamma'^\mu = M\gamma^\mu M^\dagger$ or $\gamma^\mu = M^\dagger\gamma'^\mu M$. So

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \Rightarrow (iM^\dagger\gamma'^\mu M\partial_\mu - m)\psi = 0$$
$$\Rightarrow (i\gamma'^\mu\partial_\mu - m)M\psi = 0 \Rightarrow (i\gamma'^\mu\partial_\mu - m)\psi' = 0,$$

where $\psi' = M\psi$.

Lorentz transformation properties of the Dirac equation

Firstly we want to show that the Dirac equation preserves its form in different frames. Eventually we shall be able to use the transformation properties to show that the Dirac equation is describing particles of spin $\frac{1}{2}$.

We start with the Dirac equation for $\psi(x)$:

$$\left(i\gamma^\nu\frac{\partial}{\partial x^\nu} - m\right)\psi(x) = 0.$$

If $x'^\mu = L^\mu{}_\nu x^\nu$, we have

$$\frac{\partial}{\partial x^\nu} = \frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial}{\partial x'^\mu} = L^\mu{}_\nu\frac{\partial}{\partial x'^\mu}.$$

So we can write the Dirac equation as

$$\left(i\gamma^\nu L^\mu{}_\nu\frac{\partial}{\partial x'^\mu} - m\right)\psi(x) = 0.$$

Now defining

$$\gamma'^\mu = L^\mu{}_\nu\gamma^\nu,$$

we have

$$\left(i\gamma'^\mu\frac{\partial}{\partial x'^\mu} - m\right)\psi(x) = 0.$$

But

$$\begin{aligned}\{\gamma'^\alpha, \gamma'^\beta\} =& \{L^\alpha{}_\mu\gamma^\mu, L^\beta{}_\nu\gamma^\nu\}\\ =& L^\alpha{}_\mu L^\beta{}_\nu\{\gamma^\mu, \gamma^\nu\}\\ =& 2L^\alpha{}_\mu L^\beta{}_\nu\eta^{\mu\nu} = 2\eta^{\alpha\beta}.\end{aligned}$$

By Pauli's theorem, we must have

$$\gamma'^\mu = L^\mu{}_\nu\gamma^\nu = S(L)^{-1}\gamma^\mu S(L).$$

(N.B. $S(L)$ is *not* unitary.) So we have

$$S(L)^{-1}\left(i\gamma^\mu\frac{\partial}{\partial x'^\mu} - m\right)S(L)\psi(x) = 0.$$

So if $\psi(x)$ satisfies the Dirac equation, so does

$$\begin{aligned} \psi'(x') = S(L)\psi(x) &= S(L)\psi(L^{-1}x'), \\ \text{where} \quad L^{\mu}{}_{\nu}\gamma^{\nu} &= S(L)^{-1}\gamma^{\mu}S(L), \end{aligned}$$

and where $x'^{\mu} = L^{\mu}{}_{\nu}x^{\nu}$. We interpret $\psi'(x')$ as the Lorentz-transformed version of $\psi(x)$. Note that $S(L)$ is only defined up to a scalar factor, which doesn't change the state.

L near 1

For L near the identity, we have

$$\begin{aligned} L^{\mu}{}_{\nu} &= \delta^{\mu}{}_{\nu} + \epsilon^{\mu}{}_{\nu}, \\ \text{where} \quad \epsilon_{\mu\nu} + \epsilon_{\nu\mu} &= 0. \end{aligned}$$

Then let

$$S(L) = 1_4 + i\epsilon^{\mu\nu}\Sigma_{\mu\nu}, \quad \text{where} \quad \Sigma_{\mu\nu} = -\Sigma_{\nu\mu}.$$

From

$$L^{\mu}{}_{\nu}\gamma^{\nu} = S(L)^{-1}\gamma^{\mu}S(L),$$

we have

$$\begin{aligned} \gamma^{\mu} + \epsilon^{\mu}{}_{\nu}\gamma^{\nu} &= \left(1 - i\epsilon^{\lambda\rho}\Sigma_{\lambda\rho}\right)\gamma^{\mu}\left(1 + i\epsilon^{\lambda\rho}\Sigma_{\lambda\rho}\right) + O(\epsilon^2), \\ \text{so} \quad \epsilon^{\mu}{}_{\nu}\gamma^{\nu} &= i\epsilon^{\lambda\rho}\left(\gamma^{\mu}\Sigma_{\lambda\rho} - \Sigma_{\lambda\rho}\gamma^{\mu}\right), \\ \text{i.e.} \quad \tfrac{1}{2}\epsilon^{\lambda\rho}(\delta^{\mu}{}_{\lambda}\gamma_{\rho} - \delta^{\mu}{}_{\rho}\gamma_{\lambda}) &= i\epsilon^{\lambda\rho}[\gamma^{\mu}, \Sigma_{\lambda\rho}], \\ \text{So} \quad \tfrac{1}{2}(\delta^{\mu}{}_{\lambda}\gamma_{\rho} - \delta^{\mu}{}_{\rho}\gamma_{\lambda}) &= i[\gamma^{\mu}, \Sigma_{\lambda\rho}], \end{aligned}$$

since the equation holds for any antisymmetric ϵ. The equation is satisfied by (see Homework 3)

$$\begin{aligned} \Sigma_{\lambda\rho} &= -\frac{i}{8}[\gamma_{\lambda}, \gamma_{\rho}] \\ &= -\frac{1}{4}\sigma_{\lambda\rho} \quad \text{where} \quad \sigma_{\lambda\rho} = \frac{i}{2}[\gamma_{\lambda}, \gamma_{\rho}], \\ \text{so we have} \quad S(L) &= 1 - \frac{i}{4}\epsilon^{\mu\nu}\sigma_{\mu\nu}. \end{aligned}$$

Angular momentum

To understand the angular momentum of ψ, consider taking for L a small rotation through an angle θ about an axis $\mathbf{n}$. Then

$$L^{\mu}{}_{0} = L^{0}{}_{\mu} = 0$$

$$\delta\mathbf{x} = -\,\mathbf{x}\times\mathbf{n}\theta$$

$$\Rightarrow L^{i}{}_{j}x^{j} = x^{i} - \epsilon_{ijk}x^{j}n^{k}\theta$$

$$\epsilon^{i}{}_{j} = -\,\epsilon_{ijk}n^{k}\theta = -\epsilon^{ij}$$

$$\text{So}\quad S(L) = 1 - \frac{i}{4}\epsilon_{ijk}n^{k}\theta\sigma^{ij}$$

$$= 1 + \frac{1}{8}\epsilon_{ijk}n^{k}[\gamma^{i},\gamma^{j}]\theta.$$

$$\text{Now}\quad [\gamma^{i},\gamma^{j}] = -2i\epsilon_{ijl}\begin{pmatrix}\sigma^{l} & 0\\ 0 & \sigma^{l}\end{pmatrix}$$

in the given representation, so (using $\epsilon_{ijk}\epsilon_{ijl} = 2\delta_{kl}$)

$$S(L) = 1 - \frac{i}{2}n^{k}\begin{pmatrix}\sigma^{k} & 0\\ 0 & \sigma^{k}\end{pmatrix}\theta$$

$$= 1 - i\theta\mathbf{n}.\mathbf{S},$$

$$\text{where}\quad S^{k} = \frac{1}{2}\begin{pmatrix}\sigma^{k} & 0\\ 0 & \sigma^{k}\end{pmatrix},$$

$$\text{and since}\quad [\tfrac{1}{2}\sigma^{i},\tfrac{1}{2}\sigma^{j}] = i\epsilon_{ijk}\tfrac{1}{2}\sigma^{k},$$

$$\text{we have}\quad [S^{i},S^{j}] = i\epsilon_{ijk}S^{k}.$$

The e-values of S^{3} are $\frac{1}{2}$, $-\frac{1}{2}$, $\frac{1}{2}$, $-\frac{1}{2}$. Consider a scalar field $\phi(x)$. In a Lorentz-transformed frame we have

$$\phi'(x') = \phi(x) = \phi(L^{-1}x'),\quad \text{or equivalently}\quad \phi'(x) = \phi(L^{-1}x).$$

We have

$$(Lx)^{i} = x^{i} - \epsilon_{ijk}x^{j}n^{k}\theta,$$

$$\text{so}\quad (L^{-1}x)^{i} = x^{i} + \epsilon_{ijk}x^{j}n^{k}\theta,$$

$$\text{and so}\quad \phi'(x) = \phi(x^{i} + \epsilon_{ijk}x^{j}n^{k}\theta)$$

$$= \phi(x^{i}) + \epsilon_{ijk}x^{j}n^{k}\theta\frac{\partial\phi}{\partial x^{i}} + O(\theta^{2})$$

$$= \phi(x) - \theta\mathbf{n}.(\mathbf{x}\times\nabla)\phi$$

$$= \phi(x) - i\theta\mathbf{n}.(\mathbf{x}\times\mathbf{p})\phi$$

$$= (1 - i\theta\mathbf{n}.\mathbf{L})\phi(x).$$

$$\text{So}\quad \delta\phi = -i\theta\mathbf{n}.\mathbf{L}\phi,$$

i.e. the generator of the change in ϕ corresponding to a rotation is the component of $\mathbf{L}$ in that direction. Now for the Dirac field $\psi(x)$,

$$\begin{aligned}\psi'(x) =&S(L)\psi(L^{1}x)\\ =&(1-i\theta\mathbf{n}.\mathbf{S})(1-i\theta\mathbf{n}.\mathbf{L})\psi(x)\\ =&(1-i\theta\mathbf{n}.\mathbf{J})\psi(x) \quad \text{where}\mathbf{J}=\mathbf{L}+\mathbf{S}.\end{aligned}$$

The angular momentum is defined to be the generator of rotations, so here $\mathbf{J} = \mathbf{L} + \mathbf{S}$ is the angular momentum. We have $[L_i, S_j] = 0$, and both $\mathbf{L}$ and $\mathbf{S}$ satisfy the angular momentum conservation relations. $\mathbf{S}$ has to do with the way the components of ψ are mixed by a rotation and is "internal". We identify $\mathbf{S}$ with the spin. We have $(S^i)^2 = \frac{1}{4}$ for each i and so $\mathbf{S}^2 = \frac{3}{4} = \frac{1}{2}(\frac{1}{2}+1)$. So this corresponds to spin $\frac{1}{2}$.

Parity

So far we have dealt only with $\mathcal{L}_+^\uparrow$. Now consider

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

For $S(P)$ we need

$$\begin{aligned}S(P)^{-1}\gamma^0 S(P) =&\gamma^0,\\ S(P)^{-1}\gamma^i S(P) =&-\gamma^i.\end{aligned}$$

These equations are satisfied by

$$S(P) = \gamma^0 = S(P)^{-1}$$

or by $S(P) = -\gamma^0$. We have

$$\psi'(ct, \mathbf{x}) = \eta\gamma^0\psi(ct, -\mathbf{x}), \quad \eta = \pm 1.$$

η is called the **intrinsic parity** of the particle and is totally irrelevant until you consider systems in which the types/numbers of particles change.

Probability current and bilinear forms

To give a probabilistic interpretation to the Dirac wave-function ψ we must construct a conserved current j^μ. We have

$$\begin{aligned}(i\gamma^\mu\partial_\mu - m)\psi =& 0\\ \psi^\dagger(-i\gamma^{\mu\dagger}\partial_\mu - m) =& 0.\end{aligned}$$

Using $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$, we have

$$-i\psi^\dagger\gamma^0\gamma^\mu\partial_\mu - m\psi^\dagger\gamma^0 = 0.$$

We define the Dirac adjoint $\overline{\psi} = \psi^\dagger\gamma^0$. So we have

$$\begin{aligned}\overline{\psi}(i\gamma^\mu\partial_\mu + m) =& 0\\ \Rightarrow i\overline{\psi}\gamma^\mu\partial_\mu\psi + i\overline{\psi}\gamma^\mu\partial_\mu\psi =& 0\\ \text{So} \quad j^\mu = \overline{\psi}\gamma^\mu\psi \Rightarrow \partial_\mu j^\mu = 0&\end{aligned}$$

and j^{mu} is our conserved current. Then

$$\begin{aligned}\rho = j^0 = \overline{\psi}\gamma^0\psi =& \psi^\dagger(\gamma^0)^2\psi = \psi^\dagger\psi\\ =& \sum_{\alpha=1}^{4}|\psi_\alpha|^2,\end{aligned}$$

which is positive definite as a probability density should be. Moreover, for $j^\mu = \overline{\psi}\gamma^\mu\psi$ to be satisfactory it must be a Lorentz vector, $j'^{mu}(x') = L^\mu{}_\nu j^\nu(x)$. Now

$$\begin{aligned}j'^{mu}(x') =& \psi^\dagger(x)S(L)^\dagger\gamma^0\gamma^\mu S(L)\psi(x)\\ \text{and since} \quad S(L)^{-1}\gamma^\mu S(L) =& L^\mu{}_\nu\gamma^\nu\\ \text{we have} \quad S(L)^\dagger\gamma^{\mu\dagger}S(L)^{\dagger-1} =& L^\mu{}_\nu\gamma^{\nu\dagger}\\ \text{so, using} \quad \gamma^{\mu\dagger} =& \gamma^0\gamma^\mu\gamma^0\\ \text{we have} \quad S(L)^\dagger\gamma^0\gamma^\mu\gamma^0 S(L)^{\dagger-1} =& L^\mu{}_\nu\gamma^0\gamma^\nu\gamma^0\\ \text{or} \quad \gamma^0 S(L)^\dagger\gamma^0\gamma^\mu\gamma^0 S(L)^{\dagger-1}\gamma^0 =& L^\mu{}_\nu\gamma^\nu = S(L)^{-1}\gamma^\mu S(L),\\ \text{and so} \quad S(L)\gamma^0 S(L)^\dagger\gamma^0\gamma^\mu\gamma^0 S(L)^{\dagger-1}\gamma^0 S(L)^{-1} =& \gamma^\mu\\ \text{Defining} \quad X =& S(L)\gamma^0 S(L)^\dagger\gamma^0\\ \text{we have} \quad X\gamma^\mu X^{-1} =& \gamma^\mu.\end{aligned}$$

So, from Pauli's Theorem we have

$$X = S(L)\gamma^0 S(L)^\dagger\gamma^0 = k1_4.$$

Taking the Hermitian conjugate (and using $\gamma^{0\dagger} = \gamma^0$)) we have

$$\gamma^0 S(L)\gamma^0 S(L)^\dagger = k^* 1_4,$$

and then multiplying by γ^0 on the left and on the right, and using $(\gamma^0)^2 = 1$, we have

$$S(L)\gamma^0 S(L)^\dagger \gamma^0 = k^* 1_4,$$

so we must have $k = k^*$. We also have

$$|\det S(L)|^2 = k^4,$$

and since (not proved) $\det S(L) = 1$ we have $k = \pm 1$. For $L \in \mathcal{L}_+^\uparrow$ we have $k = +1$. So

$$S(L)^\dagger \gamma^0 = \gamma^0 S(L)^{-1},$$

and

$$\begin{aligned} j'^\mu(x') =& \psi^\dagger(x)\gamma^0 S(L)^\dagger \gamma^\mu S(L)\psi(x) \\ =& L^\mu{}_\nu \overline{\psi(x)}\gamma^\nu \psi(x) \end{aligned}$$

so that j^μ *is* a vector.

We now define

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3.$$

We can show

$$\begin{aligned} (\gamma^5)^2 = 1, \quad \gamma^{5\dagger} =& \gamma^5 \\ \text{and} \quad \{\gamma^5, \gamma^\mu\} =& 0. \end{aligned}$$

In the representation we are using,

$$\gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It can be shown that

$$\begin{aligned} S(L)^{-1}\gamma^5 S(L) =& (\det L)\gamma^5 \\ =& \gamma^5 \quad \text{for} \quad L \in \mathcal{L}_+^\uparrow. \end{aligned}$$

It is also easy to show that

$$S(P)^{-1}\gamma^5 S(P) = -\gamma^5.$$

We can then show that under a Lorentz transformation

$$\begin{aligned}
\overline{\psi'(x')}\psi'(x') =&\overline{\psi(x)}S(L)^{-1}S(L))\psi(x) = \overline{\psi(x)}\psi(x),\\
\overline{\psi'(x')}\gamma^5\psi'(x') =&\overline{\psi(x)}S(L)^{-1}\gamma^5 S(L)\psi(x) = \overline{\psi(x)}\gamma^5\psi(x)\\
\overline{\psi'(x')}\gamma^\mu\psi'(x') =&L^\mu{}_\nu\overline{\psi(x)}\gamma^\nu\psi(x),\\
\overline{\psi'(x')}\gamma^5\gamma^\mu\psi'(x') =&L^\mu{}_\nu\overline{\psi(x)}\gamma^5\gamma^\nu\psi(x),
\end{aligned}$$

while under a parity transformation

$$\begin{aligned}
\overline{\psi'(x')}\psi'(x') =&\overline{\psi(x)}S(P)^{-1}S(P))\psi(x) = \overline{\psi(x)}\psi(x),\\
\overline{\psi'(x')}\gamma^5\psi'(x') =&\overline{\psi(x)}S(P)^{-1}\gamma^5 S(P)\psi(x) = -\overline{\psi(x)}\gamma^5\psi(x)\\
\overline{\psi'(x')}\gamma^\mu\psi'(x') =&P^\mu{}_\nu\overline{\psi(x)}\gamma^\nu\psi(x),\\
\overline{\psi'(x')}\gamma^5\gamma^\mu\psi'(x') =&P^\mu{}_\nu\overline{\psi(x)}\gamma^5\gamma^\nu\psi(x).
\end{aligned}$$

Quantities with the above transformation properties are called respectively scalar, pseudoscalar, vector and axial (or pseudo) vector.

Solutions of the Dirac equation

Consider the plane wave $\psi(x) = u(p)e^{-ip.x}$. It will satisfy the Dirac equation

$$\begin{aligned}
(i\gamma.\partial - m)\psi =&0\\
\text{if}\quad (\gamma.p - m)u(p) =&0.
\end{aligned}$$

Multiplying by $(\gamma.p + m)$, we find

$$\begin{aligned}
(\gamma.p - m)(\gamma.p - m)u(p) =&0\\
\Rightarrow \tfrac{1}{2}\{\gamma^\mu, \gamma^{nu}\}(p_\mu p_\nu - m^2)u(p) =&0\\
\Rightarrow (p^2 - m^2)u(p) =&0 \Rightarrow p^2 = m^2\\
\Rightarrow p^0 = \pm\sqrt{\mathbf{p}^2 + m^2}&.
\end{aligned}$$

So we still seem to have $-$ve energy solutions. It is conventional for reasons which will become apparent later to reverse the direction of the momentum for the $-$ve energy states. So we have

$$\begin{aligned}
&+\text{ve energy} &&u(p)e^{-ip.x} &&p^0 = \sqrt{\mathbf{p}^2 + m^2}\\
&-\text{ve energy} &&v(p)e^{ip.x} &&p^0 = \sqrt{\mathbf{p}^2 + m^2}.
\end{aligned}$$

It is straightforward Fourier analysis to show that these are a complete set of solutions for the Dirac equation.

We can explicitly construct $u(p)$: In our representation,

$$\gamma.p - m = \begin{pmatrix} p^0 - m & -\sigma.\mathbf{p} \\ \sigma.\mathbf{p} & -p^0 - m \end{pmatrix}$$

where each entry is a 2×2 block. Now write

$$u(p) = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \quad \text{where} \quad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \quad \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}.$$

Then

$$(\gamma.p - m)u(p) = 0$$

becomes

$$\begin{aligned} (p^0 - m)\xi &= \sigma.\mathbf{p}\eta, \\ \sigma.\mathbf{p}\xi &= (p^0 + m)\eta. \end{aligned}$$

Given ξ, define

$$\begin{aligned} \eta &= \frac{\sigma.\mathbf{p}}{p^0 + m}\xi \\ \text{then} \quad \sigma.\mathbf{p}\eta &= \frac{(\sigma.\mathbf{p})^2}{p^0 + m}\xi \\ \text{Now} \quad \sigma^i\sigma^j p_i p_j &= \mathbf{p}^2 \quad (\sigma^i\sigma^j = \delta_{ij} + i\epsilon_{ijk}\sigma^k) \\ \text{So} \quad \sigma.\mathbf{p}\eta &= \frac{\mathbf{p}^2}{p^0 + m}\xi \\ &= \frac{(p^0)^2 - m^2}{p^0 + m}\xi \\ &= (p^0 - m)\xi \end{aligned}$$

so we have a consistent solution. The general form of $u(p)$ is

$$u(p) = N(p)\begin{pmatrix} \chi \\ \frac{\sigma.\mathbf{p}}{p^0+m}\chi \end{pmatrix}.$$

We can get a basic set by choosing two 2-spinors χ_r (r=1,2) such that $\chi_r^\dagger\chi_s = \delta_{rs}$. Then we have

$$\begin{aligned} \psi_p^{(r)}(x) &= e^{-ip.x}u_r(p) \\ &= e^{-ip.x}\begin{pmatrix} \chi_r \\ \frac{\sigma.\mathbf{p}}{p^0+m}\chi_r \end{pmatrix} N(p) \end{aligned}$$

is a basic set of +ve energy states, r being the "spin label". Then

$$
\begin{aligned}
< \psi_p^{(r)}, \psi_{p'}^{(s)} >=& \int \overline{\psi_p^{(r)}} \gamma^0 \psi_{p'}^{(s)} d^3\mathbf{x} = \int \psi_p^{(r)\dagger} \gamma^0 \gamma^0 \psi_{p'}^{(s)} d^3\mathbf{x} \\
=& \int \psi_p^{(r)\dagger} \psi_{p'}^{(s)} d^3\mathbf{x} \\
=& e^{i(p^0 - p'^0)x^0} \int e^{-i(\mathbf{p}-\mathbf{p}').\mathbf{x}} d^3\mathbf{x} N(p)^* N(p') \left\{ \chi_r^\dagger \chi_s + \chi_r^\dagger \frac{\sigma.\mathbf{p}\sigma.\mathbf{p}'}{(p^0+m)(p'^0+m)} \chi_s \right\} \\
=& (2\pi)^3 \delta(\mathbf{p}-\mathbf{p}') e^{-i(p^0-p'^0)x^0} N(p)^* N(p') \left\{ \chi_r^\dagger \chi_s + \chi_r^\dagger \frac{\sigma.\mathbf{p}\sigma.\mathbf{p}'}{(p^0+m)(p'^0+m)} \chi_s \right\}.
\end{aligned}
$$

Now

$$\delta(\mathbf{p}-\mathbf{p}') f(\mathbf{p}, \mathbf{p}') = \delta(\mathbf{p}-\mathbf{p}') f(\mathbf{p}, \mathbf{p}).$$

This is because

$$\int d^3\mathbf{p}' \delta(\mathbf{p}-\mathbf{p}') f(\mathbf{p}, \mathbf{p}') = \int d^3\mathbf{p}' \delta(\mathbf{p}-\mathbf{p}') f(\mathbf{p}, \mathbf{p}) = f(\mathbf{p}, \mathbf{p}),$$

and functions containing δ-functions are actually defined by integrals like this. So we can set $\mathbf{p}' = \mathbf{p}$ and correspondingly $p'^0 = p^0$, obtaining (using $(\sigma.\mathbf{p})^2 = \mathbf{p}^2$)

$$
\begin{aligned}
< \psi_p^{(r)}, \psi_{p'}^{(s)} >=& (2\pi)^3 \delta(\mathbf{p}-\mathbf{p}') |N(p)|^2 \chi_r^\dagger \chi_s \left\{ 1 + \frac{|\mathbf{p}|^2}{(p^0+m)^2} \right\} \\
\text{Now} \quad (p^0+m)^2 + |\mathbf{p}|^2 =& (p^0)^2 + 2mp^0 + m^2 + (p^0)^2 - m^2 = 2p^0(p^0+m) \\
\text{So} \quad < \psi_p^{(r)}, \psi_{p'}^{(s)} >=& \delta_{rs} \frac{2p^0}{p^0+m} \delta(\mathbf{p}-\mathbf{p}') |N(p)|^2 (2\pi)^3.
\end{aligned}
$$

Recall with $\phi_p(x) = e^{-ip.x}$ we have

$$< \phi_p, \phi_{p'} >= i \int \phi_p(x)^* \partial^0 \phi_{p'}(x) d^3\mathbf{x} = 2p^0 (2\pi)^3 \delta(\mathbf{p}-\mathbf{p}').$$

So choose $N(p) = \sqrt{p^0 + m}$ so that

$$< \psi_p^{(r)}, \psi_{p'}^{(s)} >= \delta_{rs} 2p^0 (2\pi)^3 \delta(\mathbf{p}-\mathbf{p}').$$

We now have

$$\overline{u_r(p)} u_s(p) = 2m\delta_{rs}.$$

(See Homework 5.) A general solution of the Dirac equation is

$$\psi(x) = \int \frac{d^3\mathbf{p}}{2p^0} \sum_{r=1}^{2} \left\{ u_r(p) e^{-ip.x} f_r^+(p) + v_r(p) e^{ip.x} f_r^-(p) \right\}.$$

Negative energy states and the hole interpretation

The Dirac equation has +ve definite probability density unlike the KG equation, but it still has −ve energy states.

Spin $\frac{1}{2}$ particles are fermions and we can only have at most one fermion in each state (the Pauli exclusion principle).

Dirac supposed that the vacuum corresponded not to all states empty, but to having all of the −ve energy states full (the **sea** of −ve energy electrons).If a positive energy state is full, we observe it as a positive energy particle of mass m and charge $-e$. If a −ve energy state is unoccupied, we observe it as a (positive energy) particle of mass m, charge $+e$–the antiparticle of the electron, or **positron**.

This picture predicts electron-positron annihilation and pair production.

Because you need to fill all the −ve energy states and talk about multi-particle systems to make sense of the Dirac equation, it doesn't really describe a single particle.

2. Quantisation of Free Fields

Finite-dimensional systems

First we review the mechanics of a system with a finite number of degrees of freedom, n say. In Lagrangian mechanics such a system is described by n generalised co-ordinates $q_1 \dots q_n$ and n generalised velocities $\dot{q}_1 \dots \dot{q}_n$. The equations of motion can be derived from a variational principle

$$\delta \int_{t_1}^{t_2} dt L(q, \dot{q}, t) = 0$$

subject to $q_r(t_1) = q_r^{(1)}$, $q_r(t_2) = q_r^{(2)}$, $1 \leq r \leq n$. This leads to the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0.$$

If a particular co-ordinate q_r is absent from L, then $p_r = \frac{\partial L}{\partial \dot{q}_j}$ is constant. In general $p_r = \frac{\partial L}{\partial \dot{q}_j}$ is called the momentum conjugate to q_r. If $\frac{\partial L}{\partial q_r} = 0$, we say q_r is ignorable.

If L has no explicit time dependence then by multiplying by $\dot{q}_j$ and summing over j we can deduce

$$\dot{q}_j p_j - L = \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L = \text{constant}.$$

$$\left[\frac{d}{dt}\left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L\right) = \ddot{q}_j \frac{\partial L}{\partial \dot{q}_j} + \dot{q}_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}\dot{q}_j - \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j = 0.\right]$$

This is the energy conservation equation.

The Hamiltonian is defined by

$$H = \dot{q}_j p_j - L(q, \dot{q}, t).$$

We regard $H \equiv H(q,p,t)$. We need to "invert" $p_r = \frac{\partial L}{\partial \dot{q}_j}$ to get $\dot{q}_r \equiv \dot{q}_r(q,p,t)$. We may deduce the Hamiltonian equations of motion:

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \qquad \dot{p}_j = -\frac{\partial H}{\partial q_j}.$$

(N.B some care using the chain rule is required, since derivatives of L are implicitly defined with q or $\dot{q}$ constant, while derivatives of H are implicitly defined with q or p constant.) If $H \equiv H(q,p,t)$ has no explicit t-dependence, i.e. $H = H(q,p)$, then

$$\frac{dH}{dt} = \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i = 0,$$

i.e. H is conserved. The corresponding quantum-mechanical system is obtained by inventing operators q_i, p_i such that

$$[q_i, p_i] = i\hbar\delta_{ij} = i\hbar\{q_i, p_j\},$$

where $\{,\}$ is the Poisson bracket, defined for two classical dynamical variables α, β by

$$\{\alpha, \beta\} = \frac{\partial \alpha}{\partial q_r}\frac{\partial \beta}{\partial p_r} - \frac{\partial \alpha}{\partial p_r}\frac{\partial \beta}{\partial q_r}.$$

Then for any two quantum variables

$$[\alpha, \beta] = i\hbar\{\alpha, \beta\} + O(\hbar^2).$$

In the co-ordinate representation,

$$p_j = -i\hbar\frac{\partial}{\partial q_j}.$$

In the Schrödinger representation the state vectors vary with time as specified by the Schrödinger equation, e.g.

$$i\hbar\frac{\partial \psi}{\partial t} = H\psi,$$

and the dynamical variables $\leftrightarrow$ constant operators. To pass to the Heisenberg picture, we introduce a unitary operator defined by

$$i\hbar\frac{\partial U}{\partial t} = HU.$$

(If H is constant then $U = e^{-\frac{iHt}{\hbar}}$.) The Heisenberg picture wave function is given by

$$\psi_H = U^\dagger \psi$$

and is constant in time. We define the Heisenberg picture operators by

$$\alpha_H = U^\dagger \alpha U.$$

THen

$$\begin{aligned} <\psi, \alpha\psi> &= <U\psi_H, \alpha U\psi_H> \\ &= <\psi_H, U^\dagger \alpha U \psi_H> \\ &= <\psi_H, \alpha_H \psi> . \end{aligned}$$

Moreover α_H satisfies the equation of motion

$$i\hbar\frac{\partial \alpha_H}{\partial t} = [\alpha_H, H_H].$$

In fact the classical equations of motion may be written in the analogous form

$$\frac{d\alpha}{dt} = \{\alpha, H\}.$$

Classical Field Theories

A classical field $\phi(\mathbf{x}, t)$ may be regarded as corresponding to one degree of freedom at each point in space, so that $\mathbf{x}$ in $\phi(\mathbf{x}, t) \leftrightarrow i$ in q_i. For a classical field theory the Lagrangian is a functional of the fields $\phi(\mathbf{x}, t)$:

$$L[\phi] = \int \mathcal{L}(\phi, \partial^\mu \phi) d^3\mathbf{x},$$

and the action is

$$\int_{t_1}^{t_2} dt L[\phi] = \int d^4x \mathcal{L}(\phi, \partial^\mu \phi).$$

We obtain the equation of motion by equating to zero the variation of the action corresponding to variations in ϕ satisfying

$$\delta\phi = 0 \quad \text{at} \quad t = t_1, t_2.$$

Then

$$\delta \int \mathcal{L}(\phi, \partial^\mu \phi) d^4x = \int \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} \delta(\partial^\mu \phi) \right\} d^4x$$

$$(\delta(\partial^\mu \phi = \partial^\mu \delta\phi.)$$

$$= \int \delta\phi \left\{ \frac{\partial \mathcal{L}}{\partial \phi} - \partial^\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} \right] \right\} d^4x$$

$$+ \int \partial^\mu \left[\delta\phi \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} \right] d^4x.$$

So as $\delta\phi$ is independent for each $(\mathbf{x}, t)$,

$$\partial^\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} \right] - \frac{\partial \mathcal{L}}{\partial \phi} = 0.$$

E.g., if $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2$, the equation of motion becomes $(\partial^2 + m^2)\phi = 0$, i.e. the KG equation.

We introduce a generalised momentum

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}.$$

The Hamiltonian density is defined by

$$\mathcal{H} = \dot{\phi}(x)\pi(x) - \mathcal{L},$$

then

$$H = \int d^3\mathbf{x} \dot{\phi}(\mathbf{x}, t)\pi(\mathbf{x}, t) - L = \int \mathcal{H} d^3\mathbf{x}.$$

For the KG field $\pi = \dot{\phi}$ and

$$\mathcal{H} = \tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}m^2\phi^2.$$

The total energy $\int \mathcal{H} d^3\mathbf{x}$ should be conserved and the zero components of some 4-vector P^μ. To construct this, we introduce the energy-momentum tensor

$$T^{\mu\nu} = \partial^\mu \phi \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} - \mathcal{L}\eta^{\mu\nu}.$$

Then we have

$$P^\mu = \int T^{\mu 0} d^3\mathbf{x}$$

and

$$H = P^0 = \int T^{00} d^3\mathbf{x}.$$

$T^{\mu\nu}$ is conserved in the sense that

$$\begin{aligned}
\partial_\nu T^{\mu\nu} =& 0. \\
[\partial_\nu T^{\mu\nu} =& \partial^\mu \phi \partial_\nu \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \right\} + \partial_\nu \partial^\mu \phi \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \\
& - \eta^{\mu\nu} \left\{ \frac{\partial \mathcal{L}}{\partial \phi} + \frac{\partial \mathcal{L}}{\partial(\partial_\lambda \phi)} \partial_\nu \partial_\lambda \phi \right\} \\
=& 0
\end{aligned}$$

using the equations of motion.]

The conservation of $T^{\mu\nu}$ implies $P^\mu = \int T^{\mu 0} d^3\mathbf{x}$ transforms as a 4-vector and is time-independent (just as for $j^{mu}(x)$ earlier).

P^μ is the energy-momentum vector.

Quantisation of the K-G field (real scalar field)

The Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}m^2\phi^2$$

leads to the K-G equation

$$(\partial^2 + m^2)\phi = 0.$$

The generalised momentum is given by

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}.$$

We can write

$$\mathcal{L} = \tfrac{1}{2}\dot{\phi}^2 - \tfrac{1}{2}(\nabla\phi)^2 - \tfrac{1}{2}\phi^2.$$

The analogues of the commutation relations

$$[q_i, p_j] = i\hbar\delta_{ij}, \quad [q_i, q_j] = [p_i, p_j] = 0$$

are

$$\begin{aligned}
[\phi(\mathbf{x}, t), \pi(\mathbf{x}', t)] =& i\hbar\delta(\mathbf{x} - \mathbf{x}'),\\
[\phi(\mathbf{x}, t), \phi(\mathbf{x}', t)] =& 0,\\
[\pi(\mathbf{x}, t), \pi(\mathbf{x}', t)] =& 0.
\end{aligned}$$

Note that these are all at the same time t. These are called *equal time canonical commutation relations (CCRs).* We always assume from now on that we are in the Heisenberg picture (without explicitly writing a subscript H). The equations of motion should take the form

$$i\hbar\dot{\alpha} = [\alpha, H].$$

We have

$$H = \int \mathcal{H} d^3x = \mathcal{H} = \int [\tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}m^2\phi^2]d^3x.$$

To see that the CCRs are appropriate, consider

$$\begin{aligned}
[\phi(\mathbf{x}, t), H] =& [\phi(\mathbf{x}, t), \int [\tfrac{1}{2}\pi(\mathbf{x}', t)^2 + \tfrac{1}{2}(\nabla\phi(\mathbf{x}', t)^2 + \tfrac{1}{2}m^2\phi(\mathbf{x}', t)^2]d^3x']\\
=& \tfrac{1}{2}\int d^3\mathbf{x}' \left\{[\phi(\mathbf{x}), \pi(\mathbf{x}')^2] + [\phi(\mathbf{x}), (\nabla'\phi(\mathbf{x}'))^2] + m^2[\phi(\mathbf{x}), \phi(\mathbf{x}')^2]\right\}.
\end{aligned}$$

$$[\phi(\mathbf{x}), \phi(\mathbf{x}')] = 0 \Rightarrow [\phi(\mathbf{x}), \nabla'\phi(\mathbf{x}')] = 0.$$

$$\begin{aligned}
\text{So} \quad [\phi(\mathbf{x}), H] =& \tfrac{1}{2}\int \{\pi(\mathbf{x}')[\phi(\mathbf{x}), \pi(\mathbf{x}')] + [\phi(\mathbf{x}), \phi(\mathbf{x}')]\pi(\mathbf{x}')\}\\
=& i\hbar\int \pi(\mathbf{x}')\delta(\mathbf{x} - \mathbf{x}')d^3\mathbf{x}'\\
=& i\hbar\pi(\mathbf{x}) = i\hbar\dot{\phi}(\mathbf{x}) \quad \text{OK.}
\end{aligned}$$

(Note we have suppressed t for brevity here.) We also have

$$\begin{aligned}
[\pi(\mathbf{x},t),H] =& -\left[\int d^3\mathbf{x}'\left\{\tfrac{1}{2}\pi(\mathbf{x}')^2+\tfrac{1}{2}(\nabla\phi(\mathbf{x}')^2+\tfrac{1}{2}m^2\phi(\mathbf{x}')^2\right\},\pi(\mathbf{x})\right]\\
=& -\tfrac{1}{2}\int d^3\mathbf{x}'\left\{\nabla'\phi(\mathbf{x}')[\nabla'\phi(\mathbf{x}'),\pi(\mathbf{x})]+[\nabla'\phi(\mathbf{x}'),\pi(\mathbf{x})]\nabla'\phi(\mathbf{x}')\right\}\\
& -\tfrac{1}{2}m^2\int d^3\mathbf{x}'\left\{\phi(\mathbf{x}')[\phi(\mathbf{x}'),\pi(\mathbf{x})]+[\phi(\mathbf{x}'),\pi(\mathbf{x})]\phi(\mathbf{x}')\right\}\\
=& -i\hbar\int d^3\mathbf{x}'\left\{\nabla'\phi(\mathbf{x}').\nabla'\delta(\mathbf{x}-\mathbf{x}')+m^2\phi(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')\right\}\\
=& i\hbar\int d^3\mathbf{x}'\left\{(\nabla')^2\phi(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')-m^2\phi(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')\right\}\\
=& i\hbar\left\{\nabla^2\phi(\mathbf{x})-m^2\phi(\mathbf{x})\right\}\\
=& i\hbar\ddot{\phi}(\mathbf{x}) \quad \text{using K-G equation}\\
=& i\hbar\dot{\pi}(\mathbf{x})
\end{aligned}$$

as required. So the equations of motion are verified.

To see the connection between the quantised field and its particle interpretation, we need to look at the Fourier transformed field. We derived this earlier in the form

$$\phi(x)=\frac{1}{(2\pi)^3}\int d^3\mathbf{p}\frac{1}{2p^0}\left[f_+(\mathbf{p})e^{-ip.x}+f_-(\mathbf{p})e^{ip.x}\right]$$

For ϕ to be hermitian we require

$$\begin{aligned}
f_-(\mathbf{p}) =& [f_+(\mathbf{p})]^\dagger = a^\dagger(\mathbf{p}) \quad \text{say,}\\
f_+(\mathbf{p}) =& a(\mathbf{p}).\\
\text{Then}\quad \phi(x) =& \frac{1}{(2\pi)^3}\int d^3\mathbf{p}\frac{1}{2p^0}\left[a(\mathbf{p})e^{-ip.x}+a^\dagger(\mathbf{p})e^{ip.x}\right]
\end{aligned}$$

Recall that we showed that with $\phi_p(x)=e^{-ip.x}$, we have

$$<\phi_p,\phi_{p'}>=2p^0(2\pi)^3\delta(\mathbf{p}-\mathbf{p}').$$

We therefore have

$$\begin{aligned}
<e^{-ip.x},\phi>=& a(\mathbf{p})=\int[p^0\phi(\mathbf{x},t)+i\pi(\mathbf{x},t)]e^{ip.x}d^3\mathbf{x},\\
-<e^{ip.x},\phi>=& a^\dagger(\mathbf{p})=\int[p^0\phi(\mathbf{x},t)-i\pi(\mathbf{x},t)]e^{-ip.x}d^3\mathbf{x},
\end{aligned}$$

from which we obtain using the CCRs

$$\begin{aligned}[a(\mathbf{p}),a^\dagger(\mathbf{p}')] =&2p^0\delta(\mathbf{p}-\mathbf{p}')(2\pi)^3,\\ [a(\mathbf{p}),a(\mathbf{p}')] =&0,\\ [a^\dagger(\mathbf{p}),a^\dagger(\mathbf{p}')] =&0.\end{aligned}$$

We now have to define the space on which these operators act. We start by defining the "vacuum state" such that

$$a(\mathbf{p})|0>=0 \quad \text{for all}\mathbf{p}, <0|0>=1.$$

Single particle states are defined by

$$|\mathbf{p}>=a^\dagger(\mathbf{p})|0>.$$

Then

$$\begin{aligned}<\mathbf{p}'|\mathbf{p}>=&<0|a(\mathbf{p}')a^\dagger(\mathbf{p})|0>\\ =&<0|\{a^\dagger(\mathbf{p})a(\mathbf{p}')+[a(\mathbf{p}'),a^\dagger(\mathbf{p})]|0>\\ =&<0|\{a^\dagger(\mathbf{p})a(\mathbf{p}')|0>+<0|[a(\mathbf{p}'),a^\dagger(\mathbf{p})]|0>\\ =&2p^0\delta(\mathbf{p}-\mathbf{p}')(2\pi)^3<0|0>\\ =&2p^0\delta(\mathbf{p}-\mathbf{p}')(2\pi)^3.\end{aligned}$$

Two-particle states are defined by

$$|\mathbf{p}_1,\mathbf{p}_2>=a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_1)|0>.$$

Clearly

$$|\mathbf{p}_1,\mathbf{p}_2>=|\mathbf{p}_2,\mathbf{p}_1> \quad \text{as} \quad [a(\mathbf{p}_1),a(\mathbf{p}_2)]=0.$$

It can be shown (see Homework 6) that

$$\begin{aligned}&<\mathbf{p}_1',\mathbf{p}_2'|\mathbf{p}_1,\mathbf{p}_2>=\\ &(2\pi)^6(2p_1^0)(2p_2^0)\{\delta(\mathbf{p}_1-\mathbf{p}_1')\delta(\mathbf{p}_2-\mathbf{p}_2')+\delta(\mathbf{p}_1-\mathbf{p}_2')\delta(\mathbf{p}_2-\mathbf{p}_1')\}.\end{aligned}$$

We also have

$$<0|\mathbf{p}>=<0|a^\dagger(\mathbf{p})|0>=[<0|a(\mathbf{p})|0>]^*=0$$

and similarly $\quad <\mathbf{p}|\mathbf{p}_1\mathbf{p}_2>=0.$

An n-particle state is defined by

$$|\mathbf{p}_1\mathbf{p}_2\dots\mathbf{p}_n>= a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)\dots a^\dagger(\mathbf{p}_n)|0>,$$

which is symmetric under permutations of $\mathbf{p}_1\dots\mathbf{p}_n$. It is this symmetry which is the defining characteristic of bosons and leads to wave-functions which are symmetric under interchange of particles.

We have

$$\begin{aligned}< \mathbf{p}'_1\mathbf{p}'_2\dots\mathbf{p}'_m|\mathbf{p}_1\mathbf{p}_2\dots\mathbf{p}_n > &=0 \quad \text{for} \quad m\neq n,\\ &= \sum_{\text{perms of } 1,\dots n}\prod_{r=1}^{n} 2p_r^0(2\pi)^3\delta(\mathbf{p}'_{\rho(r)}-\mathbf{p}_r).\end{aligned}$$

If

$$\begin{aligned}\mathcal{H}_n &=\text{space of} \quad n \quad \text{particle states}\\ &=\text{space spanned by} \quad \{|\mathbf{p}_1\dots\mathbf{p}_n>,\end{aligned}$$

then the total space

$$\mathcal{H}=+\mathcal{H}_n$$

which is the space of states of an indefinite number of bosons. We define a number operator

$$N=\frac{1}{(2\pi)^3}\int\frac{d^3\mathbf{p}}{2p^0}a^\dagger(\mathbf{p})a(\mathbf{p}),$$

which satisfies

$$[N,a^\dagger(\mathbf{p})]=a^\dagger(\mathbf{p})$$

and hence

$$N|\mathbf{p}_1\dots\mathbf{p}_n>= n|\mathbf{p}_1\dots\mathbf{p}_n>$$

(see Homework 6). So N counts the number of particles. We may interpret

$$N(\mathbf{p})=\frac{1}{(2\pi)^3}\frac{1}{2p^0}a^\dagger(\mathbf{p})a(\mathbf{p})$$

as the number density in momentum space, and therefore the total momentum should be given by

$$\begin{aligned}P^\mu &=\int N(\mathbf{p})p^\mu d^3\mathbf{p}\\ &=\frac{1}{(2\pi)^3}\int\frac{d^3\mathbf{p}}{2p^0}a^\dagger(\mathbf{p})a(\mathbf{p})p^\mu\end{aligned}$$

We should also be able to derive this directly from the earlier expression for the energy-momentum tensor $T^{\mu\nu}$. In particular

$$P^0 = H = \tfrac{1}{2}\int[\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2]d^3x.$$

Now

$$\begin{aligned}\int \phi^2 d^3\mathbf{x} =& \frac{1}{(2\pi)^6}\int d^3\mathbf{x}\int d^3\mathbf{p}\frac{1}{2p^0}\left[a(\mathbf{p})e^{-ip.x} + a^\dagger(\mathbf{p})e^{ip.x}\right] \\ & \int d^3\mathbf{p}'\frac{1}{2p'^0}\left[a(\mathbf{p}')e^{-ip'.x} + a^\dagger(\mathbf{p}')e^{ip'.x}\right], \\ \int \dot{\phi}^2 d^3\mathbf{x} =& \frac{1}{(2\pi)^6}\int d^3\mathbf{x}\int d^3\mathbf{p}\frac{1}{2p^0}\left[-ip^0 a(\mathbf{p})e^{-ip.x} + ip^0 a^\dagger(\mathbf{p})e^{ip.x}\right] \\ & \int d^3\mathbf{p}'\frac{1}{2p'^0}\left[-ip'^0 a(\mathbf{p}')e^{-ip'.x} + ip'^0 a^\dagger(\mathbf{p}')e^{ip'.x}\right], \\ \int (\nabla\phi)^2 d^3\mathbf{x} =& \frac{1}{(2\pi)^6}\int d^3\mathbf{x}\int d^3\mathbf{p}\frac{1}{2p^0}\left[i\mathbf{p}a(\mathbf{p})e^{-ip.x} - i\mathbf{p}a^\dagger(\mathbf{p})e^{ip.x}\right] \\ & \int d^3\mathbf{p}'\frac{1}{2p'^0}\left[i\mathbf{p}'a(\mathbf{p}')e^{-ip'.x} - i\mathbf{p}'a^\dagger(\mathbf{p}')e^{ip'.x}\right],\end{aligned}$$

and so, using

$$\int d^3\mathbf{x}e^{-ip.x}e^{-ip'.x} = (2\pi)^3 e^{-i(p^0+p'^0)}\delta(\mathbf{p}+\mathbf{p}'),$$

and $(p^0)^2 = \mathbf{p}^2 + m^2$, we find

$$\begin{aligned}P^0 =& \frac{1}{2}\frac{1}{(2\pi)^3}\int d^3\mathbf{p}\frac{1}{(2p^0)^2}\{[-(p^0)^2 + \mathbf{p}^2 + m^2]a(\mathbf{p})a(\mathbf{p}) + [-(p^0)^2 + \mathbf{p}^2 + m^2]a^\dagger(\mathbf{p})a^\dagger(\mathbf{p}) \\ &+ [(p^0)^2 + \mathbf{p}^2 + m^2](a^\dagger(\mathbf{p})a(\mathbf{p}) + a(\mathbf{p})a^\dagger(\mathbf{p}))\} \\ =& \frac{1}{2}\frac{1}{(2\pi)^3}\int d^3\mathbf{p}\frac{1}{2p^0}p^0(a^\dagger(\mathbf{p})a(\mathbf{p}) + a(\mathbf{p})a^\dagger(\mathbf{p})).\end{aligned}$$

In general we find

$$\begin{aligned}P^\mu =& \frac{1}{2}\frac{1}{(2\pi)^3}\int d^3\mathbf{p}\frac{1}{2p^0}p^\mu[a^\dagger(\mathbf{p})a(\mathbf{p}) + a(\mathbf{p})a^\dagger(\mathbf{p})] \\ =& \frac{1}{(2\pi)^3}\int d^3\mathbf{p}\frac{1}{2p^0}p^\mu[a^\dagger(\mathbf{p})a(\mathbf{p}) + \tfrac{1}{2}(2\pi)^3 2p^0\delta(0)].\end{aligned}$$

The $\delta(0)$ means that we get an infinite answer. We merely discard this infinity to get the previous result. This is not as arbitrary as it appears. The trouble merely comes from the ordering of a and $a^\dagger$ in H, which we could have adjusted freely before passing to the quantum-mechanical expression. If we'd had sufficient foresight to do this, we could have obtained the earlier expression straightaway. We need to write H so that all the annihilation operators a stand to the right of all the creation operators $a^\dagger$. This process is called **normal ordering** or Wick ordering and is denoted by colons.

Write

$$\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x)$$

where

$$\begin{aligned}
\phi^{(+)}(x) =& \frac{1}{(2\pi)^3}\int \frac{d^3\mathbf{p}}{2p^0} a(\mathbf{p}) e^{-ip.x} \\
=& \frac{1}{(2\pi)^3}\int d^4p\delta(p^2 - m^2)\theta(p^0) e^{-ip.x} a(\mathbf{p}), \\
\phi^{(-)}(x) =& \frac{1}{(2\pi)^3}\int \frac{d^3\mathbf{p}}{2p^0} a^\dagger(\mathbf{p}) e^{ip.x} \\
=& \frac{1}{(2\pi)^3}\int d^4p\delta(p^2 - m^2)\theta(p^0) e^{ip.x} a(\mathbf{p}).
\end{aligned}$$

The normal-ordered product

$$: \phi(x)^2 := [\phi^{(+)}(x)]^2 + [\phi^{(-)}(x)]^2 + 2\phi^{(-)}(x)\phi^{(+)}(x).$$

Define

$$\begin{aligned}
H =& \int d^3\mathbf{x}\frac{1}{2} : (\dot\phi^2 + (\nabla\phi)^2 + m^2\phi^2) : \\
=& \frac{1}{(2\pi)^3}\int \frac{d^3\mathbf{p}}{2p^0} a^\dagger(\mathbf{p}) a(\mathbf{p}) p^0 = \int d^3\mathbf{p} N(\mathbf{p}) p^0.
\end{aligned}$$

Can show

$$[P^\mu, \phi(x)] = -i\partial^\mu\phi(x) \quad \text{Check}$$

from which it follows that

$$e^{-ia.P}\phi(x)e^{ia.P} = \phi(x-a).$$

Arbitrary-time commutators and singular distributions

The CCRs specify thecommutators of $\phi(x_1)$, $\phi(x_2)$ if the times of the operators are the same.

Now consider $[\phi(x_1), \phi(x_2)]$ without this restriction.

The answer is a complex-valued function, not an operator.

Let

$$\begin{aligned}
[\phi(x_1),\phi(x_2)] =& i\Delta(x_1,x_2)\\
\Rightarrow e^{-ia.P}[\phi(x_1),\phi(x_2)]e^{ia.P} =& ie^{-ia.P}\Delta(x_1,x_2)e^{ia.P}\\
[e^{-ia.P}\phi(x_1)e^{ia.P},e^{-ia.P}\phi(x_2)e^{ia.P}] =& i\Delta(x_1,x_2)\\
[\phi(x_1-a),\phi(x_2-a)] =& i\Delta(x_1-a,x_2-a)=i\Delta(x_1,x_2)\\
\Rightarrow \Delta(x_1,x_2) =& \Delta(x_1-x_2,0) \quad (a=x_2).
\end{aligned}$$

With a slight change of notation,

$$\begin{aligned}
[\phi(x_1),\phi(x_2)] =& i\Delta(x_1-x_2)\\
=& [\phi^{(+)}(x_1),\phi^{(-)}(x_2)]+[\phi^{(-)}(x_1),\phi^{(+)}(x_2)]\\
=& \frac{1}{(2\pi)^3}\int\frac{d^3\mathbf{p}}{2p^0}e^{-ip.(x_1-x_2)}-\frac{1}{(2\pi)^3}\int\frac{d^3\mathbf{p}}{2p^0}e^{ip.(x_1-x_2)}.\\
\text{So}\quad \Delta(x) =& -\frac{i}{(2\pi)^3}\int\frac{d^3\mathbf{k}}{2k^0}(e^{-ik.x}-e^{ik.x}),\quad k^0=\sqrt{\mathbf{k}^2+m^2}\\
=& -\frac{i}{(2\pi)^3}\int d^4k e^{-ik.x}\epsilon(k^0)\delta(k^2-m^2)\\
\text{where}\quad \epsilon(\eta) =& +1 \quad (\eta>0)\\
=& -1 \quad (\eta<0).
\end{aligned}$$

Can show

$$\begin{aligned}
&\text{(a)}\quad (\partial^2+m^2)\Delta = 0\\
&\text{(b)}\quad \frac{\partial}{\partial x^0}\Delta = -\delta(\mathbf{x})\\
&\text{(c)}\quad \Delta(Lx) = \Delta(x)\\
&\text{(d)}\quad \Delta(x) = 0 \quad \text{for} \quad x^2<0.
\end{aligned}$$

Proofs:

(a) Get in $\int$: $(k^2 - m^2)\delta(k^2 - m^2) = 0.$

$$\text{(b)} \quad [\phi(x_1), \frac{\partial}{x_2^0}\phi(x_2)] = -i\frac{\partial \Delta}{\partial x_2^0}(x_1 - x_2)$$

$$\text{Put} \quad x_1^0 = x_2^0 = t$$

$$\Rightarrow -i\frac{\partial \Delta}{\partial x_2^0}(x_1 - x_2) = [\phi(\mathbf{x}_1, t), \phi(\mathbf{x}_2, t)] = = \delta(\mathbf{x}_1 - \mathbf{x}_2).$$

$$\text{(c)} \quad \Delta(Lx) = -\frac{i}{(2\pi)^3}\int d^4k e^{-ik.Lx}\epsilon(k^0)\delta(k^2 - m^2)$$

$$= -\frac{i}{(2\pi)^3}\int d^4k e^{-iL^{-1}k.x}\epsilon(k^0)\delta(k^2 - m^2).$$

$$\text{Write} \quad k' = L^{-1}k, \epsilon(k'^0) = \epsilon(k^0) \quad \text{for} \quad L \in \mathcal{L}_+^\uparrow$$

$$\Rightarrow \Delta(Lx) = -\frac{i}{(2\pi)^3}\int d^4k' e^{-ik'.x}\epsilon(k'^0)\delta(k'^2 - m^2)$$

$$= \Delta(x).$$

$$\text{(d)} \quad [\phi(x), \phi(0)] = 0 \quad \text{if} \quad x^0 = 0.$$

If $x^2 < 0$ there is a $L \in \mathcal{L}_+^\uparrow$ such that $Lx = x'$ where x'^0. Then

$$[\phi(x'), \phi(0)] = 0 = i\Delta(x') = i\Delta(Lx) = i\Delta(x).$$

A useful formula is

$$\Delta(x) = \frac{1}{(2\pi)^4}\int_{\mathcal{C}} \frac{d^4k}{k^2 - m^2} e^{-ik.x}.$$

The integral is defined by a contour integral in the complex k^0 plane followed by a real $d^3\mathbf{k}$ integral.

Poles at $k^0 = \pm\sqrt{\mathbf{k}^2 + m^2}$. Proof:

Collapse onto poles.

$$-\frac{2\pi i}{(2\pi)^4}\int \frac{d^3\mathbf{k}}{2k^0}\left(e^{-ik.x} - e^{ik.x}\right).$$

Again obvious that $(\partial^2 + m^2)\Delta(x) = 0$.

We can use the contour integrals to construct Green functions for

$$(\partial^2 + m^2)\phi = j$$

appropriate to various boundary conditions.

$$\begin{aligned}(\partial^2 + m^2)G(x) =& \delta(\mathbf{x}) \\ G_{\mathcal{C}} =& \frac{1}{(2\pi)^4}\int_{\mathcal{C}} \frac{d^4k}{k^2 - m^2} e^{-ik.x} \\ (\partial^2 + m^2)G_{\mathcal{C}} =& \frac{1}{(2\pi)^4}\int_{\mathcal{C}} d^4k e^{-ik.x} \\ =& \ \delta(\mathbf{x})\end{aligned}$$

if $\mathcal{C}$ is any contour from $-\infty$ to $+\infty$ near the real axis in the k^0 plane. Different choices of $\mathcal{C}$ correspond to different boundary conditions.

(a)

$$\Delta_{\text{adv}}(x) = -\frac{1}{(2\pi)^4}\int_{\mathcal{C}_{\text{adv}}} \frac{d^4k}{k^2 - m^2} e^{-ik.x}.$$

If $e^{-ik.x} \to 0$ exponentially as $k^0 \to -i\infty$ then $\Delta_{\text{adv}}(x) = 0$. Write $k^0 = -i\rho \Rightarrow e^{-\rho x^0} \to 0$ exponentially as $\rho \to \infty$ if $x^0 > 0$. So

$$\begin{aligned}\Delta_{\text{adv}}(x) =& 0 \quad \text{for} \quad x^0 > 0 \\ (\partial^2 + m^2)\Delta_{\text{adv}}(x) =& \delta(\mathbf{x}).\end{aligned}$$

To solve

$$\begin{aligned}(\partial^2 + m^2)\phi =& j \\ \phi \to \phi_{\text{out}}(x), \quad & x^0 \to +\infty \\ \text{where} \quad (\partial^2 + m^2)\phi_{\text{out}} =& 0 \\ \phi(x) =& \phi_{\text{out}}(x) + \int j(x')\Delta_{\text{adv}}(x - x')d^4x'.\end{aligned}$$

(b)

$$\Delta_{\rm ret}(x) = -\frac{1}{(2\pi)^4}\int_{\mathcal{C}_{\rm ret}} \frac{d^4k}{k^2-m^2} e^{-ik.x}$$

$$\Delta_{\rm ret}(x) = 0 \quad \text{for} \quad x^0 < 0$$

$$(\partial^2 + m^2)\Delta_{\rm ret}(x) = \delta(\mathbf{x}).$$

To solve

$$(\partial^2 + m^2)\phi = j$$

$$\phi \to \phi_{\rm in}(x), \quad x^0 \to -\infty$$

$$\text{where} \quad (\partial^2 + m^2)\phi_{\rm in} = 0$$

$$\phi(x) = \phi_{\rm in}(x) + \int j(x')\Delta_{\rm ret}(x-x')d^4x'.$$

(c)

$$\Delta_F(x) = \frac{1}{(2\pi)^4}\int_{\mathcal{C}_F} \frac{d^4k}{k^2-m^2} e^{-ik.x}.$$

On the right, $k^0 = \sqrt{\mathbf{k}^2+m^2} + i\epsilon$. Equivalently, displace pole downwards to $k^0 = \sqrt{\mathbf{k}^2+m^2} - i\epsilon$.

$$\Delta_F(x) = \frac{1}{(2\pi)^4}\int_{\mathcal{C}_F} \frac{d^4k}{k^2-m^2+i\epsilon} e^{-ik.x}.$$

Implicitly we take $\epsilon \to 0$. For $x^0 > 0$ as in the discussion of Δ_{adv} we may close the contour towards $k^0 \to -i\infty$.

$$\begin{aligned}\Delta_F(x) &= -\frac{1}{(2\pi)^4}\int_{\mathcal{C}+}\frac{d^4k}{k^2-m^2}e^{-ik.x}\\ &= -\frac{i}{(2\pi)^4}\int\frac{d^3\mathbf{k}}{2k^0}e^{-ik.x}\\ &= -i[\phi^{(+)}(x),\phi^{(-)}(0)].\end{aligned}$$

For $x^0 < 0$ we may close the contour towards $k^0 \to +i\infty$.

$$\begin{aligned}\Delta_F(x) &= -\frac{1}{(2\pi)^4}\int_{\mathcal{C}-}\frac{d^4k}{k^2-m^2}e^{-ik.x}\\ &= -\frac{i}{(2\pi)^4}\int\frac{d^3\mathbf{k}}{2k^0}e^{ik.x}\\ &= -i[[\phi^{(+)}(0),\phi^{(-)}(x)].\end{aligned}$$

Now

$$\begin{aligned}< 0|\phi(x)\phi(0)|0 > &= < 0|\phi^{(+)}(x)\phi^{(-)}(0)|0 >\\ &= < 0|[\phi^{(+)}(x),\phi^{(-)}(0)]|0 >\\ &= i\Delta(x) \quad \text{if} \quad x^0 > 0.\\ < 0|\phi(0)\phi(x)|0 > &= < 0|[\phi^{(+)}(0),\phi^{(-)}(x)]|0 >\\ &= \Delta(x) \quad \text{if} \quad x^0 < 0.\end{aligned}$$

Define the **time-ordered product** by

$$\begin{aligned}T\{\phi(x_1),\phi(x_2)\} &= \phi(x_1)\phi(x_2) \quad (x_1^0 > x_2^0)\\ &= \phi(x_2)\phi(x_1) \quad (x_1^0 < x_2^0).\end{aligned}$$

Then

$$\begin{aligned}< 0|T\{\phi(x),\phi(0)\}|0 > &= i\Delta_F(x),\\ < 0|T\{\phi(x_1),\phi(x_2)\}|0 > &= i\Delta_F(x_1-x_2).\end{aligned}$$

The Dirac field

We saw that the K-G field described bosons, because of the CCRs

$$\begin{aligned}[a(\mathbf{p}), a^\dagger(\mathbf{p}')] =& 2p^0\delta(\mathbf{p}-\mathbf{p}')(2\pi)^3,\\ [a(\mathbf{p}), a(\mathbf{p}')] =& 0,\\ [a^\dagger(\mathbf{p}), a^\dagger(\mathbf{p}')] =& 0.\\ |\mathbf{p},\mathbf{p}'>=& |\mathbf{p}',\mathbf{p}> .\end{aligned}$$

To obtain fermions we would need

$$|\mathbf{p},\mathbf{p}'>= -|\mathbf{p}',\mathbf{p}>$$

which would result if $a^\dagger(\mathbf{p})a^\dagger(\mathbf{p}') = -a^\dagger(\mathbf{p})a^\dagger(\mathbf{p}')$, i.e.

$$\{a^\dagger(\mathbf{p}), a^\dagger(\mathbf{p}')\} = 0.$$

It turns out that the Dirac equation requires such anti-commutation relations to ensure positive energy, and therefore describes fermions. The Dirac equation can be obtained from the Lagrange density

$$\begin{aligned}\mathcal{L} =& \overline{\psi}(i\gamma_\mu\partial^\mu - m)\psi\\ \text{or}\quad \mathcal{L}' =& \tfrac{1}{2}\overline{\psi}i\gamma_\mu\partial^\mu\psi - m\overline{\psi}\psi.\end{aligned}$$

These two Lagrange densities differ by a total derivative $i\frac{1}{2}\partial^\mu(\overline{\psi}\gamma_\mu\psi)$, and so give the same equations of motion (the second one is hermitian). The generalised momentum is given by

$$\frac{\partial\mathcal{L}}{\partial\dot{\psi}_\alpha} = i(\overline{\psi}\gamma^0)_\alpha = i\psi^\dagger.$$

Note that this does not involve $\dot{\psi}$! We postulate the **anti**-commutation relations

$$\{\psi_\alpha(\mathbf{x},t), i\psi_\beta(\mathbf{x}',t)\} = i\delta(\mathbf{x}-\mathbf{x}')\delta_{\alpha\beta},$$

which can be written

$$\{\psi_\alpha(\mathbf{x},t), \overline{\psi}_\beta(\mathbf{x}',t)\} = \delta(\mathbf{x}-\mathbf{x}')\gamma^0_{\alpha\beta}.$$

We supplement this with the assumption that

$$\{\psi_\alpha(\mathbf{x},t), \psi_\beta(\mathbf{x}',t)\} = \{\overline{\psi}_\alpha(\mathbf{x},t), \overline{\psi}_\beta(\mathbf{x}',t)\} = 0.$$

We shall see shortly that this leads to positive energy. Then

$$H = \int d^3\mathbf{x}[-i\overline{\psi}\gamma.\nabla\psi + m\overline{\psi}\psi].$$

The CCRs should lead to the correct equations of motion. We should have

$$\begin{aligned}\dot{\psi} = [\psi_\alpha, H] =& -i\int\{\psi_\alpha(\mathbf{x}), \overline{\psi}_\beta(\mathbf{x}')\}[\gamma.\nabla\psi(\mathbf{x}')]_\beta d^3\mathbf{x}' + m\int\{\psi_\alpha(\mathbf{x}), \overline{\psi}_\beta(\mathbf{x}')\}\psi_\beta(\mathbf{x}')d^3\mathbf{x}' \\ =& -i[\gamma^0\gamma.\nabla\psi(\mathbf{x})]_\alpha + m[\gamma^0\psi(\mathbf{x})]_\alpha\end{aligned}$$

which is indeed the Dirac equation for ψ. [We have used here

$$[\psi, BC] = \{\psi, B\}C - B\{\psi, C\}.]$$

We may rewrite H using the Dirac equation as

$$H = \int d^3\mathbf{x}\overline{\psi}i\gamma^0\partial_0\psi.$$

The energy-momentum tensor is given by

$$T^{\mu\nu} = i\overline{\psi}\gamma^\mu\partial^\nu\psi.$$

Using the general solution of the Dirac equation, we introduce annihilation and creation operators:

$$\begin{aligned}\psi(x) =& \frac{1}{(2\pi)^3}\sum_{r=1}^{2}\int\frac{d^3\mathbf{p}}{2p^0}\left\{u_r(p)e^{-ip.x}a_r(p) + v_r(p)e^{ip.x}b_r^\dagger(p)\right\} \\ \text{and so}\quad \overline{\psi}(x) =& \frac{1}{(2\pi)^3}\sum_{r=1}^{2}\int\frac{d^3\mathbf{p}}{2p^0}\left\{\overline{u}_r(p)e^{ip.x}a_r^\dagger(p) + \overline{v}_r(p)e^{-ip.x}b_r(p)\right\}.\end{aligned}$$

The anticommutation relations of the ψs and $\overline{\psi}$s lead to (see Homework 7)

$$\begin{aligned}\{a_r(\mathbf{p}), a_s^\dagger(\mathbf{p}')\} =& (2\pi)^3\delta_{rs}2p^0\delta(\mathbf{p}-\mathbf{p}'), \\ \{b_r(\mathbf{p}), b_s^\dagger(\mathbf{p}')\} =& (2\pi)^3\delta_{rs}2p^0\delta(\mathbf{p}-\mathbf{p}'), \\ \{a_r(\mathbf{p}), a_s(\mathbf{p}')\} = \{a_r^\dagger(\mathbf{p}), a_s^\dagger(\mathbf{p}')\} =& \{a_r(\mathbf{p}), b_s^\dagger(\mathbf{p}')\} = \{a_r(\mathbf{p}), b_s(\mathbf{p}')\} = 0.\end{aligned}$$

We may calculate P^μ from $T^{\mu\nu}$ (again see Homework 7):

$$P^\mu = \sum_{s=1}^{2}\frac{1}{(2\pi)^3}\int\frac{d^3\mathbf{p}}{2p^0}p^\mu\left[a_s^\dagger(\mathbf{p})a_s(\mathbf{p}) - b_s(\mathbf{p})b_s^\dagger(\mathbf{p})\right].$$

This differs by an infinite constant from

$$P^\mu = \sum_{s=1}^{2} \frac{1}{(2\pi)^3} \int \frac{d^3\mathbf{p}}{2p^0} p^\mu \left[a_s^\dagger(\mathbf{p}) a_s(\mathbf{p}) + b_s^\dagger(\mathbf{p}) b_s(\mathbf{p}) \right],$$

which we would have obtained straightaway if we had normal ordered, defining

$$P^\mu = \int : T^{\mu 0} : d^3\mathbf{x}$$

and defined

$$: bb^\dagger := -b^\dagger b, \quad : b^\dagger b := b^\dagger b,$$

etc. The vacuum state $|0>$ satisfies

$$a_r(\mathbf{p})|0>= b_r(\mathbf{p})|0>$$

for $r = 1, 2$ and for all $\mathbf{p}$. A general state is of the form

$$a_{r_1}^\dagger(\mathbf{k}_1) \ldots a_{r_m}^\dagger(\mathbf{k}_m) b_{s_1}^\dagger(\mathbf{p}_1) \ldots b_{s_n}^\dagger(\mathbf{p}_n)|0>= |\mathbf{k}_1 r_1, \ldots \mathbf{k}_m r_m; \mathbf{p}_1 s_1 \ldots \mathbf{p}_n s_n>,$$

which is interpreted as a state with fermions of momenta $\mathbf{k}_1 \ldots \mathbf{k}_m$ and helicities $r_1 \ldots r_m$ respectively, together with antifermions of momenta $\mathbf{p}_1 \ldots \mathbf{p}_n$ and helicities $s_1 \ldots s_n$ respectively.

3. Interacting fields

We now wish to study interactions between fields, and self-interactions. Then the equation of motion will be derived from a Lagrangian density

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I,$$

where $\mathcal{L}$ is a sum of free Lagrangian densities for each field concerned, i.e. $\frac{1}{2}(\partial\phi_r)^2 - \frac{1}{2}m_r^2\phi_r^2$ for each scalar field ϕ_r, and $\overline{\psi}_r(i\gamma^\mu\partial_\mu - m_r)\psi_r$ for each fermion field. $\mathcal{L}_I$ (the interaction Lagrangian density) contains interactions between fields, and self-interactions, such as

$$-\frac{\lambda_4}{4!}\phi^4, \quad -\frac{\lambda_3}{3!}\phi^3, \quad -g_1\phi\overline{\psi}\psi, - \quad g_2\phi\gamma^5\overline{\psi}\psi, \quad -g(\overline{\psi}\gamma^\mu\psi)(\overline{\psi}\gamma_\mu\psi).$$

We could also have higher powers of ϕ, vector fields, derivatives etc.

A simple explicit example:

$$\begin{aligned}\mathcal{L}_0 =& \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}m^2\phi^2\\ \mathcal{L}_I =& -\frac{\lambda_3}{3!}\phi^3 - \frac{\lambda_4}{4!}\phi^4\end{aligned}$$

.

From this we obtain the CCRs. Firstly

$$\pi(\mathbf{x},t) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}(\mathbf{x},t)$$

provided that $\mathcal{L}_I$ contains no time derivatives. Then the standard CCRs

$$\begin{aligned}[\phi(\mathbf{x},t),\pi(\mathbf{x}',t)] =& i\hbar\delta(\mathbf{x}-\mathbf{x}'),\\ [\phi(\mathbf{x},t),\phi(\mathbf{x}',t)] =& 0,\\ [\pi(\mathbf{x},t),\pi(\mathbf{x}',t)] =& 0\end{aligned}$$

still hold (i.e. lead to the correct equations of motion) provided $\mathcal{L}_I$ is just linear in $\partial^\mu\phi$. However, to expand $\phi(\mathbf{x},t)$ in terms of $a^\dagger(\mathbf{p})$ and $a(\mathbf{p})$, we needed the free-field equation of motion; but the equations of motion are now complicated and non-linear.

The interaction picture

Usually in elementary quantum mechanics we use the Schrödinger picture, where the wave-function (or state vector) is time-varying, and the observables are time-independent.

On the other hand, in discussing free-field quantum field theory we have used the Heisenberg picture in which the state vectors are fixed and the dynamical variables evolve according to the Heisenberg equation

$$i\hbar\frac{\partial\phi}{\partial t} = [\phi, H], \quad \text{etc.}$$

In interacting quantum field theory (treated *perturbatively*) it is convenient to adopt a compromise between the two pictures. The interaction picture is defined as follows:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$$

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L}$$

$$= (\pi\dot{\phi} - \mathcal{L}_0) - \mathcal{L}_I$$

$$= \mathcal{H}_0 + \mathcal{H}_I, \quad \text{where} \quad \mathcal{H}_I = -\mathcal{L}_I \equiv V.$$

$$\text{So} \quad H = H_0 + H_I,$$

$$\text{where} \quad H_0 = \int d^3\mathbf{x}[\tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}m^2\phi^2],$$

$$H_I = -\int d^3\mathbf{x}\mathcal{L}_I.$$

Suppose at $t = 0$,

$$\psi_H(0) = \psi_S(0),$$

$$\psi_S(t) = U(t)\psi_H(0)$$

$$\text{where} i\hbar\frac{dU}{dt} = HU.$$

$$\alpha_H = U^{-1}\alpha U.$$

If H is constant,

$$U = e^{-\frac{iHt}{\hbar}}.$$

Define

$$\psi_I(t) = e^{\frac{iH_0t}{\hbar}}\psi_S(t)$$

$$i\hbar\frac{d\psi_I}{dt} = e^{\frac{iH_0t}{\hbar}}H_I e^{\frac{-iH_0t}{\hbar}}\psi_I$$

$$= \tilde{H}_I\psi_I.$$

$$\alpha_I = e^{\frac{iH_0t}{\hbar}}\alpha_S e^{\frac{-iH_0t}{\hbar}}$$

$$i\hbar\frac{d\alpha_I}{dt} = [\alpha_I, H_0].$$

Thus in the interaction picture dynamical variables move as though they were free. In particular ϕ will satisfy the free-field equations. The complicatd additional t-dependence resulting from H_I is shifted onto the states.

The field ϕ_I is called the "in" field because it evolves according to the **free** equation from what it was before interaction.

The S-matrix

The usefulness of the interaction picture (perturbation theory) depends on the interaction being negligible at very early and very late times. Then the Heisenberg fields are free fields and we can analyse the state of the system in terms of the particle states of the field ("in"field at $t \to -\infty$, "out" field at $t \to +\infty$).

$$\begin{aligned} \phi_H \to& \phi_{\text{in}} \sim a_{\text{in}}, a^\dagger_{\text{in}} \quad \text{as} \quad t \to -\infty, \\ \phi_H \to& \phi_{\text{out}} \sim a_{\text{out}}, a^\dagger_{\text{out}} \quad \text{as} \quad t \to -\infty. \end{aligned}$$

It is an assumption that ϕ_H behaves like a free field as $t \to \pm\infty$ (sometimes called the asymptotic assumption).

Consider the relation between the Heisenberg and interaction pictures.

$$\begin{aligned} \phi_H(t) =& e^{\frac{iHt}{\hbar}} \phi_S e^{\frac{-iHt}{\hbar}}, \\ \phi_{\text{in}}(t) =& e^{\frac{iH_0 t}{\hbar}} \phi_S e^{\frac{-iH_0 t}{\hbar}}, \\ \Rightarrow \phi_H(t) =& U(t)^\dagger \phi_{\in}(t) U(t) \\ \text{where} \quad U(t) = e^{\frac{iH_0 t}{\hbar}} e^{\frac{-iHt}{\hbar}} & \\ \text{so} \quad i\hbar \frac{dU}{dt} =& e^{\frac{iH_0 t}{\hbar}} (H - H_0) e^{\frac{-iHt}{\hbar}} \\ =& e^{\frac{iH_0 t}{\hbar}} H_I e^{\frac{-iH_0 t}{\hbar}} U(t) \\ =& \tilde{H}_I U(t), \\ \text{where} \quad \tilde{H}_I =& e^{\frac{iH_0 t}{\hbar}} H_I e^{\frac{-iH_0 t}{\hbar}}. \end{aligned}$$

For large t we have

$$\begin{aligned} \phi_H(t) = \phi_{\text{out}}(t) =& \frac{1}{(2\pi)^3} \int d^3\mathbf{p} \frac{1}{2p^0} \left[a_{\text{out}}(\mathbf{p}) e^{-ip.x} + a^\dagger_{\text{out}}(\mathbf{p}) e^{ip.x} \right] \\ \phi_{\text{in}}(t) =& \frac{1}{(2\pi)^3} \int d^3\mathbf{p} \frac{1}{2p^0} \left[a_{\text{in}}(\mathbf{p}) e^{-ip.x} + a^\dagger_{\text{in}}(\mathbf{p}) e^{ip.x} \right], \\ \phi_{\text{out}}(t) =& U(t)^\dagger \phi_{\text{in}}(t) U(t). \end{aligned}$$

Let

$$S = U(\infty),$$

then

$$a_{\text{out}} = S^\dagger a_{\text{in}} S, \quad a^\dagger_{\text{out}} = S^\dagger a^\dagger_{\text{in}} S,$$
$$a_{\text{in}} = S a_{\text{out}} S^\dagger, \quad a^\dagger_{\text{in}} = S a^\dagger_{\text{out}} S^\dagger.$$

Thus if $|\text{out}>$ denotes a certain state built up from the creation operators of the out-field in a certain way, we can write $|\text{out}>= S^\dagger|\text{in}>$ where $|\text{in}>$ is the state made up out of the creation operators of the in-field in the same way (assuming $S|0>=0$). For example,

$$\begin{aligned}|\mathbf{k}_1, \mathbf{k}_2, \text{out}>=&a^\dagger_{\text{out}}(\mathbf{k}_1)a^\dagger_{\text{out}}(\mathbf{k}_2)|0>\\ =&(S^\dagger a_{\text{in}}(\mathbf{k}_1)S)(S^\dagger a_{\text{in}}(\mathbf{k}_2)S)|0>\\ =&S^\dagger a_{\text{in}}(\mathbf{k}_1)a_{\text{in}}(\mathbf{k}_2)|0>\\ =&S^\dagger|\mathbf{k}_1, \mathbf{k}_2, \text{in}> .\end{aligned}$$

Suppose we have a certain state in the Heisenberg picture which has a certain structure in terms of in-field creation operators. To see what it looks like after scattering we have to expand in terms of out-field states,

$$|\text{in}>= \sum c_{\text{out}'}|\text{out}'> .$$

The probability of a transition to a given out state $|\text{out}'>$ is given by $|<\text{out}'|\text{in}>|^2$ where $|\text{out}'>= S^\dagger|\text{in}'>$, $|\text{in}'>$ being the state created in the same way as $|\text{out}'>$ but by in-field operators. So

$$\begin{aligned}<\text{out}'|\text{in}>=&<\text{in}'|S|\text{in}>\\ =&<\text{out}'|S|\text{out}>\end{aligned}$$

where $|\text{out}>$ is the state created in the same way as $|\text{in}>$ but by out-field operators.

To be specific, the probability of a transition from the state of two particles coming in with momenta $\mathbf{p}_1$, $\mathbf{p}_2$ to two particles coming out with momenta $\mathbf{k}_1$, $\mathbf{k}_2$ is given by

$$\begin{aligned}<\mathbf{k}_1, \mathbf{k}_2, \text{out}|\mathbf{p}_1, \mathbf{p}_2, \text{in}>=&<\mathbf{k}_1, \mathbf{k}_2, \text{in}|S|\mathbf{p}_1, \mathbf{p}_2, \text{in}>\\ =&<\mathbf{k}_1, \mathbf{k}_2, \text{out}|S|\mathbf{p}_1, \mathbf{p}_2, \text{out}> .\end{aligned}$$

We need an expression for the S-matrix in terms of the in-fields. We have

$$i\hbar\frac{dU}{dt} = \tilde{H}_I U,$$

where $\tilde{H}_I$ is the interaction Hamiltonian calculated in terms of in-fields. Integrating from $-\infty$ to t, and using $U(-\infty) = 1$,

$$U(t) = 1 - \frac{i}{\hbar}\int_{-\infty}^{t} dt' \tilde{H}_I(t')U(t').$$

Now substitute this solution into its own RHS, and iterate.

$$\begin{aligned} U(t) =& 1 - \frac{i}{\hbar}\int_{-\infty}^{t} dt_1 \tilde{H}_I(t_1) + \left(-\frac{i}{\hbar}\right)^2 \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \tilde{H}_I(t_1)\tilde{H}_I(t_2)U(t_2) \\ =& 1 + \sum_{n=1}^{\infty}\left(-\frac{i}{\hbar}\right)^n \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \ldots \int_{-\infty}^{t_{n-1}} dt_n \tilde{H}_I(t_1)\tilde{H}_I(t_n)\ldots \tilde{H}_I(t_n). \end{aligned}$$

For instance, if $\mathcal{L} = -\frac{\lambda}{3!}\phi^3$, this is an expansion in powers of λ; λ is called the coupling constant. The integral in the second term can be written as

$$\begin{aligned} & \frac{1}{2}\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \tilde{H}_I(t_1)\tilde{H}_I(t_2) + \frac{1}{2}\int_{-\infty}^{t} dt_2 \int_{-\infty}^{t_2} dt_1 \tilde{H}_I(t_2)\tilde{H}_I(t_1) \\ =& \frac{1}{2}\int_{-\infty}^{t} dt_1 dt_2 T\{\tilde{H}_I(t_1)\tilde{H}_I(t_2)\}. \end{aligned}$$

In general,

$$U(t) = 1 + \sum_{n=1}^{\infty}\left(-\frac{i}{\hbar}\right)^n \frac{1}{n!}\int_{-\infty}^{t} dt_1 dt_2 \ldots dt_n T\{\tilde{H}_I(t_1)\tilde{H}_I(t_n)\ldots \tilde{H}_I(t_n)\},$$

where

$$T\{\alpha(t_1), \dots \alpha(t_n)\} = \alpha(t_{\rho(1)}) \dots \alpha(t_{\rho(n)}),$$

where ρ is a permutation of $1, \dots n$ such that $t_{\rho(i)} \geq t_{\rho(j)}$ if $i < j$. Thus

$$S = U(\infty) = 1 + \sum_{n=1}^{\infty} \left(-\frac{i}{\hbar}\right)^n \frac{1}{n!} \int_{-\infty}^{\infty} dt_1 dt_2 \dots dt_n T\{\tilde{H}_I(t_1)\tilde{H}_I(t_n)\dots \tilde{H}_I(t_n)\}.$$

Since

$$H_I(t) = -L_I(t) = -\int d^3\mathbf{x}\mathcal{L}_I,$$

we have

$$S = 1 + \sum_{n=1}^{\infty} \left(\frac{i}{\hbar}\right)^n \frac{1}{n!} \int_{-\infty}^{\infty} d^4x_1 d^4x_2 \dots d^4x_n T\{\mathcal{L}_I(x_1)\mathcal{L}_I(x_n)\dots \mathcal{L}_I(x_n)\}.$$

Here $\mathcal{L}$ is calculated from the in-fields. We can write this in shorthand form as

$$S = T\exp\left(\frac{i}{\hbar}\int_{-\infty}^{\infty} d^4x\mathcal{L}_I(x)\right).$$

Feynman diagrams in position space

The thing we need to calculate is:

$$\int_{-\infty}^{\infty} d^4x_1 d^4x_2 \dots d^4x_n T\{\mathcal{L}_I(x_1)\mathcal{L}_I(x_n)\dots \mathcal{L}_I(x_n)\},$$

(where we assume that $\mathcal{L}$ is normal ordered) or rather its matrix elements between particle states of the in-field, e.g. the contribution to

$$< \mathbf{p}'_1 \dots \mathbf{p}'_m, \text{in}|S|\mathbf{p}_1 \dots \mathbf{p}_n, \text{in} > .$$

To do this, we use Wick's Theorem. The simplest version of Wick's Theorem is

$$T\{\phi(x_1)\phi(x_2)\} = i\Delta_F(x_1 - x_2) + :\phi(x_1)\phi(x_2):$$

$$
\begin{aligned}
T\{\phi(x_1)\phi(x_2)\} =& \phi(x_1)\phi(x_2)\theta(x_1^0 - x_2^0) + \phi(x_2)\phi(x_1)\theta(x_2^0 - x_1^0) \\
=& \phi^{(+)}(x_1)\phi^{(+)}(x_2) + \phi^{(-)}(x_1)\phi^{(+)}(x_2) \\
& + \phi^{(-)}(x_2)\phi^{(+)}(x_1) + \phi^{(-)}(x_1)\phi^{(-)}(x_2) \\
& + [\phi^{(+)}(x_1), \phi^{(-)}(x_2)]\theta(x_1^0 - x_2^0) + [\phi^{(+)}(x_2), \phi^{(-)}(x_1)]\theta(x_2^0 - x_1^0) \\
=& i\Delta(x_1 - x_2) + :\phi(x_1)\phi(x_2): .
\end{aligned}
$$

$$
\begin{aligned}
T\{\phi(x_1)\dots\phi(x_n)\} = &:\phi(x_1)\dots\phi(x_n): \\
& + \frac{1}{2}\frac{1}{(n-2)!}\sum_\rho i\Delta_F(x_{\rho(1)} - x_{\rho(2)}) :\phi(x_{\rho(3)})\dots\phi(x_{\rho(n)}): \\
& + \frac{1}{2^2}\frac{1}{2!}\frac{1}{(n-4)!}\sum_\rho i\Delta_F(x_{\rho(1)} - x_{\rho(2)}) i\Delta_F(x_{\rho(3)} - x_{\rho(4)}) \\
& :\phi(x_{\rho(5)})\dots\phi(x_{\rho(n)}): + \dots .
\end{aligned}
$$

The sums are over all permutations ρ of $1, \dots n$. Note $\Delta_F(x_1 - x_2) = \Delta_F(x_2 - x_1)$. E.g.

$$
T\{\phi(x_1)\phi(x_2)\phi(x_3)\} =: \phi(x_1)\phi(x_2)\phi(x_3) : + i\Delta(x_1 - x_2)\phi(x_3) + 2\text{terms}.
$$

$$
\begin{aligned}
T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\} = &:\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4): \\
& + i\Delta(x_1 - x_2) :\phi(x_3)\phi(x_4) + 5\text{terms} \\
& + i\Delta(x_1 - x_2) i\Delta(x_3 - x_4) + 2\text{terms}.
\end{aligned}
$$

A typical problem:

2nd order contribution:

$$
A = \int d^4x_1 d^4x_2 \frac{i^2}{2} < \mathbf{p}_3\mathbf{p}_4 | T\{\mathcal{L}_I(x_1)\mathcal{L}_I(x_2)\} | \mathbf{p}_1\mathbf{p}_2 >,
$$

where

$$\mathcal{L}_I(x) = \frac{\lambda}{3!} : \phi(x)^3 : .$$

We need to calculate

$$T\left\{\frac{:\phi(x_1)\phi(x_1)\phi(x_1)::\phi(x_2)\phi(x_2)\phi(x_2):}{3!\qquad\qquad 3!}\right\}.$$

The calculation of

$$T\{:\phi(y_1)\ldots\phi(y_n)::\phi(z_1)\ldots\phi(z_n):\}$$

is the same as

$$T\{\phi(y_1)\ldots\phi(y_n)\phi(z_1)\ldots\phi(z_n)\}$$

but missing out those terms involving $\Delta_F(y_i - y_j)$ or $\Delta_F(z_i - z_j)$.

$$\begin{aligned}
T\left\{\frac{:\phi(x_1)\phi(x_1)\phi(x_1):}{3!}\frac{:\phi(x_2)\phi(x_2)\phi(x_2):}{3!}\right\} =& \frac{1}{3!}\frac{1}{3!} : \phi(x_1)^3\phi(x_2)^3 : \\
&+ \frac{1}{2!}\frac{1}{2!} i\Delta_F(x_1 - x_2) : \phi(x_1)^2\phi(x_2)^2 : \\
&+ \frac{1}{2}[i\Delta_F(x_1 - x_2)]^2 : \phi(x_1)\phi(x_2) : \\
&+ \frac{1}{6}[i\Delta_F(x_1 - x_2)]^3.
\end{aligned}$$

Now

$$\begin{aligned}
\phi^{(+)}(x)|\mathbf{p}_1 > =& \phi^{(+)}(x)a^\dagger(\mathbf{p}_1)|0> \\
=& \frac{1}{(2\pi)^3}\int \frac{d^3\mathbf{p}}{2p^0} a(\mathbf{p})e^{-ip.x}a^\dagger(\mathbf{p}_1)|0> \\
=& \frac{1}{(2\pi)^3}\int \frac{d^3\mathbf{p}}{2p^0} e^{-ip.x}[a(\mathbf{p}), a^\dagger(\mathbf{p}_1)]|0> \\
\phi^{(+)}(x)|\mathbf{p}_1 > =& e^{-ip_1.x}|0> .
\end{aligned}$$

$$\text{So} \quad < \mathbf{p}_1|\phi^{(-)}(x) = < 0|e^{ip_1.x}.$$

$$\phi^{(+)}(x_1)\phi^{(+)}(x_2)|\mathbf{p}_1 > = \phi^{(+)}(x_1)e^{-ip_1.x_2}|0> = 0.$$

$$\phi^{(+)}(x_1)\ldots\phi^{(+)}(x_n)|\mathbf{p}_1\ldots\mathbf{p}_m > = 0 \quad \text{if} \quad n > m.$$

$$\begin{aligned}
\phi^{(+)}(x_1)\phi^{(+)}(x_2)|\mathbf{p}_1\mathbf{p}_2 > =& \phi^{(+)}(x_1)\phi^{(+)}(x_2)a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)|0> \\
=& e^{-i(p_1.x_1+p_2.x_2)}|0> + e^{-i(p_1.x_2+p_2.x_1)}|0> .
\end{aligned}$$

$$\phi^{(+)}(x)|\mathbf{p}_1\mathbf{p}_2 > = e^{-ip_1.x}|\mathbf{p}_2> + e^{-ip_2.x}|\mathbf{p}_1> .$$

So

$$
\begin{aligned}
&< \mathbf{p}_3, \mathbf{p}_4 | T\Big\{ \frac{:\phi(x_1)\phi(x_1)\phi(x_1):}{3!} \frac{:\phi(x_2)\phi(x_2)\phi(x_2):}{3!} \Big\} | \mathbf{p}_1\mathbf{p}_2 >= 0 \\
&+ e^{-i(p_1+p_2).x_2} e^{i(p_3+p_4).x_1} i\Delta_F(x_1 - x_2) + e^{i(p_4-p_1).x_2} e^{i(p_3-p_2).x_1} i\Delta_F(x_1 - x_2) \\
&+ e^{i(p_4-p_2).x_2} e^{i(p_3-p_1).x_1} i\Delta_F(x_1 - x_2) + \quad 3 \text{ terms with } \quad x_1 \leftrightarrow x_2 \\
+ \quad & \frac{1}{2} < \mathbf{p}_4\mathbf{p}_2 > e^{ip_3.x_1} e^{-ip_1.x_2} [i\Delta_F(x_1 - x_2)]^2 + 3 \text{ similar terms} \\
&\qquad\qquad\qquad +4 \text{ terms with } \quad x_1 \leftrightarrow x_2 \\
&+ \frac{1}{6} [i\Delta_F(x_1 - x_2)]^3 < \mathbf{p}_3, \mathbf{p}_4 | \mathbf{p}_1\mathbf{p}_2 > .
\end{aligned}
$$

$$
\begin{aligned}
< \mathbf{p}_3, \mathbf{p}_4 | \mathbf{p}_1\mathbf{p}_2 > &= < 0 | a(\mathbf{p}_3) a(\mathbf{p}_4) a^\dagger(\mathbf{p}_1) a^\dagger(\mathbf{p}_2) | 0 > \\
&= < \mathbf{p}_4 | \mathbf{p}_2 >< \mathbf{p}_3 | \mathbf{p}_1 > + < \mathbf{p}_4 | \mathbf{p}_1 >< \mathbf{p}_3 | \mathbf{p}_2 > .
\end{aligned}
$$

So

$$
\begin{aligned}
A =& (i\lambda)^2 \int e^{-i(p_1+p_2).x_2} e^{i(p_3+p_4).x_1} i\Delta_F(x_1 - x_2) d^4x_1 d^4x_2 \\
&+ (i\lambda)^2 \int e^{i(p_4-p_1).x_2} e^{i(p_3-p_2).x_1} i\Delta_F(x_1 - x_2) d^4x_1 d^4x_2 \\
&+ (i\lambda)^2 \int e^{i(p_4-p_2).x_2} e^{i(p_3-p_1).x_1} i\Delta_F(x_1 - x_2) d^4x_1 d^4x_2 \\
&+ \frac{(i\lambda)^2}{2} < \mathbf{p}_4 | \mathbf{p}_2 > \int e^{ip_3.x_1} e^{-ip_1.x_2} [i\Delta_F(x_1 - x_2)]^2 d^4x_1 d^4x_2 + 3 \text{ similar terms} \\
&+ \frac{(i\lambda)^2}{2} \frac{1}{6} \int [i\Delta_F(x_1 - x_2)]^3 d^4x_1 d^4x_2 (< \mathbf{p}_4 | \mathbf{p}_2 >< \mathbf{p}_3 | \mathbf{p}_1 > + < \mathbf{p}_4 | \mathbf{p}_1 >< \mathbf{p}_3 | \mathbf{p}_2 >).
\end{aligned}
$$

These terms can be represented diagramatically:

We can write down rules for reconstructing A from the diagrams: Label the vertices by "dummy variables" $x_1, \ldots x_n$. For each diagram we write down:

a factor of $(-i\lambda)$ for each vertex.

a propagator $i\Delta_F(x_i - x_j)$ for each line joining x_i to x_j.

a factor $e^{-ip.x}$ for each incoming external line of momentum p arriving at vertex x.

a factor $e^{ip.x}$ for each outgoing external line of momentum p exiting at vertex x.

a factor $(2\pi)^3 2p_i^0 \delta(\mathbf{p}_i - \mathbf{p}_j)$ for each line going straight through.

a combinatorial symmetry factor $\frac{1}{|G|}$ where $|G|$ is the order of the symmetry group of the diagram keeping external lines fixed.

For $-\frac{\lambda}{n!}\phi^n$ we have n lines meeting at a vertex.

Finally integrate d^4x_i for all i.

To obtain $< \mathbf{p}'_1 \ldots \mathbf{p}'_s|S|\mathbf{p}_1 \ldots \mathbf{p}_r >$ we sum the integrals corresponding to all diagrams of the form:

Feynman rules in momentum space

Usually one uses rules in momentum space. Technique is to substitute Fourier integrals for the propagator functions. Have

$$\Delta_F(x) = \frac{1}{(2\pi)^4}\int \frac{e^{-ip.x}d^4p}{p^2 - m^2 + i\epsilon}$$

with the implicit limit $\epsilon \to 0$. We may then do all the x integrations.

$$\Delta(x_i - x_j) = \frac{1}{(2\pi)^4}\int \frac{d^4p_{ij}e^{ip_{ij}.x_i}e^{-p_{ij}.x_j}}{p_{ij}^2 - m^2 + i\epsilon}$$

e.g.

We regard p_{ij} as a momentum flowing from x_i to x_j. Then at a given vertex x get

$$\int d^4x e^{-i(\sum p_j).x} = (2\pi)^4\delta(\sum p_j)$$

where $\sum p_j$ = sum of momenta fed in at x (momentum out counted with a $-$ve sign) and we interpret p_{ij} as a momentum flowing internally from x_i *to* x_j. $\delta(\sum p_j)$ only contributes when $\sum p_j = 0$, i.e. when we have momentum conservation at the vertex.

$$\begin{aligned} p_1 + p_{41} =& p_{12} \\ p_{12} + p_2 =& p_{23} \\ p_{23} =& p_{34} + p_3 \\ p_{34} =& p_4 + p_{41} \\ \Rightarrow p_1 + p_2 =& p_3 + p_4 \\ p_{41} = k \Rightarrow p_{12} = k + p_1 \Rightarrow p_{23} =& k + p_1 + p_2 \Rightarrow p_{34} = k + p_1 + p_2 - p_3 \\ \Rightarrow p_{41} =& k + p_1 + p_2 - p_3 - p_4. \end{aligned}$$

The effect of the 4 δ-functions associated wth the 4 vertices is to produce one overall energy-momentum conservation δ-function.

$$\delta(\sum p_i - \sum p_f)$$

together with (in this case) expessions for the momenta in the propagators in terms of one independent loop momentum.

In general, we start with P 4-momenta if there are P propagators and we get V δ-functions, where V is the number of vertices. For a planar graph $V - P + L = 1$. Replace P integrations and V δ-functions by L integrations over independent loop momenta and 1 overall energy-momentum conserving δ-function. We also use $V - P + L = 1$ to redistribute factors of $(2\pi)^4$. We have

$$(2\pi)^{-4P}(2pi)^{4V} = (2\pi)^{4-4L}.$$

We associate $(2\pi)^4$ with $\delta(\sum p_i - \sum p_j)$, and $\frac{1}{(2\pi)^4}$ with each loop integration.

Feynman Rules in Momentum Space

$-i\lambda$ for a vertex

$\frac{i}{p^2 - m^2 + i\epsilon}$ for a propagator with momentum p flowing along it

$\frac{1}{(2\pi)^4} \int d^4k$ for each independent loop

$(2\pi)^4 \delta(\sum p_i - \sum p_f)$–overall momentum conservation.

Calculation of a T-matrix element

We define the T-matrix by

$$S = 1 + i(2\pi)^4 \delta(\sum p_i - \sum p_j) T.$$

The Feynman rules for T are the same as those for S without $(2\pi)^4\delta(\sum p_i - \sum p_j)$, and with a factor of $(-i)$. However, we consider all the "disconnected" diagrams to be comprised within the "1". They obviously do not correspond to real interactions between the particles, and in fact can be considered as quantum corrections to the Feynman propagator.

Consider the real scalar field theory with

$$\mathcal{L}_I = -\frac{\lambda}{3!} : \phi^3 :$$

Suppose the quanta of the field are called ϵ. Consider

$$\epsilon(p_1) + \epsilon(p_2) \to \epsilon(p_3) + \epsilon(p_4)$$

and calculate T to lowest non-trivial order in λ.

For two to two scattering it is useful to define the following invariants:

$$\begin{aligned} s = (p_1 + p_2)^2 =& (p_3 + p_4)^2, \\ t = (p_1 - p_4)^2 =& (p_2 - p_3)^2, \\ u = ((p_1 - p_3)^2 =& (p_4 - p_2)^2. \end{aligned}$$

s is the square of the total energy in the center of mass (CM) frame. (In the CM frame let

$$P = p_1 + p_2 \Rightarrow P^2 = P_0^2 - \mathbf{P}^2 = P_0^2$$

since $\mathbf{P} = 0$ in the CM frame by definition.) We can show (even in the case of unequal masses, with $p_i^2 = m_i^2$) that

$$s + t + u = \sum m_i^2 \quad (= 4m^2 \quad \text{for equal masses}).$$

For our 3 diagrams using the momentum space Feynman rules there is no loop momentum integration to do; we just have to enforce momentum conservation at each vertex. We have

$$< \mathbf{p}_3\mathbf{p}_4|T|\mathbf{p}_1\mathbf{p}_2 >= (-i)\left[\frac{(-i\lambda)^2 i}{s - m^2} + \frac{(-i\lambda)^2 i}{t - m^2} + \frac{(-i\lambda)^2 i}{u - m^2}\right].$$

Then in the centre-of-mass frame we have

$$p_1 = \begin{pmatrix} E \\ \mathbf{p}_1 \end{pmatrix}, \quad p_2 = \begin{pmatrix} E \\ -\mathbf{p}_1 \end{pmatrix}, \quad p_3 = \begin{pmatrix} E' \\ \mathbf{p}_3 \end{pmatrix}, \quad p_3 = \begin{pmatrix} E' \\ -\mathbf{p}_3 \end{pmatrix},$$

where

$$E = \sqrt{\mathbf{p}_1^2 + m^2}, \quad E' = \sqrt{\mathbf{p}_3^2 + m^2}.$$

But energy conservation $\Rightarrow 2E = 2E' \Rightarrow |\mathbf{p}_3| = |\mathbf{p}_1$. So

$$\begin{aligned} s =& (p_1 + p_2)^2 = 4E^2 = 4(\mathbf{p}_1^2 + m^2) \\ t =& (p_1 - p_4)^2 = -(\mathbf{p}_1 + \mathbf{p}_3)^2 = -2|\mathbf{p}_1|^2(1 + \cos\theta), \\ u =& (p_1 - p_3)^2 = -(\mathbf{p}_1 - \mathbf{p}_3)^2 = -2|\mathbf{p}_1|^2(1 - \cos\theta). \end{aligned}$$

So

$$< \mathbf{p}_3\mathbf{p}_4|T|\mathbf{p}_1\mathbf{p}_2 >=$$
$$\lambda^2 \left[\frac{1}{2|\mathbf{p}|^2(1 - \cos\theta) + m^2} + \frac{1}{2|\mathbf{p}|^2(1 + \cos\theta) + m^2} - \frac{1}{4(m^2 + |\mathbf{p}|^2) - m^2} \right].$$