

MATH425 - Quantum Field Theory

Set Work: Sheet 11

1. Using the normal-ordered energy-momentum vector for a scalar field,

$$P^\mu = \frac{1}{(2\pi)^3} \int d^3\mathbf{p} \frac{1}{2p^0} p^\mu [a^\dagger(\mathbf{p})a(\mathbf{p})],$$

and with ϕ given as usual by

$$\phi(x) = \frac{1}{(2\pi)^3} \int d^3\mathbf{p} \frac{1}{2p^0} [a(\mathbf{p})e^{-ip \cdot x} + a^\dagger(\mathbf{p})e^{ip \cdot x}],$$

show that

$$[P^\mu, \phi(x)] = -i\partial^\mu \phi(x).$$

Check that this implies that the equation

$$e^{-ia \cdot P} \phi(x) e^{ia \cdot P} = \phi(x - a)$$

is satisfied to 1st order in a . (In fact it is true to all orders.)

2. **(Revision)** Consider the expansion for $\phi(x)$ in real scalar quantum field theory

$$\phi(x) = \frac{1}{(2\pi)^3} \int d^3\mathbf{p} \frac{1}{2p^0} [a(\mathbf{p})e^{-ip \cdot x} + a^\dagger(\mathbf{p})e^{ip \cdot x}],$$

where $p^0 = \sqrt{\mathbf{p}^2 + m^2}$ and $a^\dagger(\mathbf{p})$ and $a(\mathbf{p})$ are creation and annihilation operators respectively for particles of momentum \mathbf{p} .

- (i) Verify by direct computation that

$$a^\dagger(\mathbf{p}) = \int [p^0 \phi(\mathbf{x}, t) - i\pi(\mathbf{x}, t)] e^{ip \cdot x} d^3\mathbf{x},$$

where the 4-vector $x = (t, \mathbf{x})$ and t is arbitrary.

- (ii) Assuming also the corresponding result

$$a(\mathbf{p}) = \int [p^0 \phi(\mathbf{x}, t) + i\pi(\mathbf{x}, t)] e^{-ip \cdot x} d^3\mathbf{x},$$

(which follows by hermitian conjugation) deduce the canonical commutation relations

$$\begin{aligned} [a(\mathbf{p}), a^\dagger(\mathbf{p}')] &= 2p^0 \delta(\mathbf{p} - \mathbf{p}') (2\pi)^3, \\ [a^\dagger(\mathbf{p}), a^\dagger(\mathbf{p}')] &= 0. \end{aligned}$$

3. Show that

$$\phi^{(+)}(x_1) \phi^{(+)}(x_2) |\mathbf{p}_1 \mathbf{p}_2\rangle = e^{-i(p_1 \cdot x_1 + p_2 \cdot x_2)} |0\rangle + e^{-i(p_1 \cdot x_2 + p_2 \cdot x_1)} |0\rangle$$

4. Show that

$$\langle \mathbf{p}_3 \mathbf{p}_4 | \mathbf{p}_1 \mathbf{p}_2 \rangle = \langle \mathbf{p}_4 | \mathbf{p}_2 \rangle \langle \mathbf{p}_3 | \mathbf{p}_1 \rangle + \langle \mathbf{p}_4 | \mathbf{p}_1 \rangle \langle \mathbf{p}_3 | \mathbf{p}_2 \rangle.$$

5. The Feynman propagator is defined by

$$\Delta_F(x) = \frac{1}{(2\pi)^4} \int_{\mathcal{C}_F} \frac{d^4 k}{k^2 - m^2} e^{-ik \cdot x}.$$

for an appropriate contour \mathcal{C}_F in the complex k^0 plane.

(a) Sketch the contour \mathcal{C}_F .

(b) Defining

$$\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x)$$

where

$$\phi^{(+)}(x) = \frac{1}{(2\pi)^3} \int \frac{d^3 \mathbf{p}}{2p^0} a(\mathbf{p}) e^{-ip \cdot x}$$

$$\phi^{(-)}(x) = \frac{1}{(2\pi)^3} \int \frac{d^3 \mathbf{p}}{2p^0} a^\dagger(\mathbf{p}) e^{ip \cdot x},$$

show that for $x^0 > 0$

$$\Delta_F(x) = -i[\phi^{(+)}(x), \phi^{(-)}(0)],$$

while for $x^0 < 0$

$$\Delta_F(x) = -i[[\phi^{(+)}(0), \phi^{(-)}(x)].$$

(c) Hence show that

$$\langle 0 | T\{\phi(x), \phi(0)\} | 0 \rangle = i\Delta_F(x).$$