

MATH425 - Quantum Field Theory

Set Work: Sheet 10

1. The current $j^\mu = \bar{\psi}\gamma^\mu\psi$ is conserved, i.e. satisfies $\partial_\mu j^\mu = 0$. In relativistic quantum mechanics we interpreted $\int j^0 d^3\mathbf{x}$ as the probability. Now we are going to interpret $Q = \int : j^0 : d^3\mathbf{x}$ as the total charge (electric charge if ψ describes electrons). Show that,

$$Q = \int d^3\mathbf{p} [N_a(\mathbf{p}) - N_b(\mathbf{p})],$$

where

$$N_a(\mathbf{p}) = \frac{1}{(2\pi)^3} \frac{1}{2p^0} a_r^\dagger(\mathbf{p}) a_r(\mathbf{p}),$$

and similarly for $N_b(\mathbf{p})$. We interpret $N_a(\mathbf{p})$, $N_b(\mathbf{p})$ as the number density of particles and antiparticles respectively; so this indicates that particles and antiparticles have equal and opposite charges.

2. Consider the Lagrangian density

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I,$$

with

$$\begin{aligned}\mathcal{L}_0 &= \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 \\ \mathcal{L}_I &= -\frac{\lambda_3}{3!}\phi^3 - \frac{\lambda_4}{4!}\phi^4.\end{aligned}$$

- (i) Derive the equation of motion from

$$\partial^\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} \right] - \frac{\partial \mathcal{L}}{\partial \phi} = 0.$$

- (ii) With the generalized momentum $\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$, write down the Hamiltonian defined by

$$H = \int [\pi \dot{\phi} - \mathcal{L}] d^3\mathbf{x}.$$

- (iii) Show that if we take the usual canonical commutation relations,

$$\begin{aligned}[\phi(\mathbf{x}, t), \pi(\mathbf{x}', t)] &= i\hbar \delta(\mathbf{x} - \mathbf{x}'), \\ [\phi(\mathbf{x}, t), \phi(\mathbf{x}', t)] &= 0, \\ [\pi(\mathbf{x}, t), \pi(\mathbf{x}', t)] &= 0,\end{aligned}$$

the above equation of motion is also obtained from

$$i\dot{\pi} = [\pi, H].$$