

MATH425 Quantum Field Theory Solutions 9

1(a). We have

$$\begin{aligned}
 \langle \psi_p^{(r)}, \tilde{\psi}_{p'}^{(s)} \rangle &= \int d^3\mathbf{x} \sqrt{p^0 + m} \sqrt{p'^0 + m} e^{ip \cdot x} e^{ip' \cdot x} \begin{pmatrix} \chi_r^\dagger & \chi_r^\dagger \frac{\sigma \cdot \mathbf{p}}{p^0 + m} \end{pmatrix} \begin{pmatrix} \frac{\sigma \cdot \mathbf{p}'}{p'^0 + m} \chi_s \\ \chi_s \end{pmatrix} \\
 &= \int d^3\mathbf{x} \sqrt{p^0 + m} \sqrt{p'^0 + m} e^{i(p^0 + p'^0)x^0} e^{-i(\mathbf{p} + \mathbf{p}') \cdot \mathbf{x}} \\
 &\quad \left(\chi_r^\dagger \frac{\sigma \cdot \mathbf{p}'}{p'^0 + m} \chi_s + \chi_r^\dagger \frac{\sigma \cdot \mathbf{p}}{p^0 + m} \chi_s \right) \\
 &= \sqrt{p^0 + m} \sqrt{p'^0 + m} e^{i(p^0 + p'^0)x^0} (2\pi)^3 \delta(\mathbf{p} + \mathbf{p}') \\
 &\quad \left(\chi_r^\dagger \frac{\sigma \cdot \mathbf{p}'}{p'^0 + m} \chi_s + \chi_r^\dagger \frac{\sigma \cdot \mathbf{p}}{p^0 + m} \chi_s \right) \\
 &= (2\pi)^3 (p^0 + m) e^{2ip^0} \left(-\chi_r^\dagger \frac{\sigma \cdot \mathbf{p}}{p^0 + m} \chi_s + \chi_r^\dagger \frac{\sigma \cdot \mathbf{p}}{p^0 + m} \chi_s \right) \\
 &= 0.
 \end{aligned}$$

Taking the hermitian conjugate of this result, we also have

$$\langle \tilde{\psi}_p^{(r)}, \psi_{p'}^{(s)} \rangle = 0,$$

and we can also derive (in the same way as for $\langle \psi_p^{(r)}, \psi_{p'}^{(s)} \rangle$)

$$\langle \tilde{\psi}_p^{(r)}, \tilde{\psi}_{p'}^{(s)} \rangle = \delta_{rs} 2p^0 (2\pi)^3 \delta(\mathbf{p} - \mathbf{p}').$$

Taking the scalar product of $\tilde{\psi}_p^{(s)}$ with

$$\psi(x) = \frac{1}{(2\pi)^3} \sum_{r=1}^2 \int \frac{d^3\mathbf{p}'}{2p'^0} [\psi_{p'}^{(r)}(x) a_r(\mathbf{p}') + \tilde{\psi}_{p'}^{(r)}(x) b_r^\dagger(\mathbf{p}')],$$

we find

$$\begin{aligned}
 \langle \tilde{\psi}_p^{(s)}, \psi \rangle &= \frac{1}{(2\pi)^3} \sum_{r=1}^2 \int \frac{d^3\mathbf{p}'}{2p'^0} [\langle \tilde{\psi}_p^{(s)}, \psi_{p'}^{(r)} \rangle a_r(\mathbf{p}') + \langle \tilde{\psi}_p^{(s)}, \tilde{\psi}_{p'}^{(r)} \rangle b_r^\dagger(\mathbf{p}')] \\
 &= \frac{1}{(2\pi)^3} \sum_{r=1}^2 \int \frac{d^3\mathbf{p}'}{2p'^0} \delta_{rs} 2p^0 (2\pi)^3 \delta(\mathbf{p} - \mathbf{p}') b_r^\dagger(\mathbf{p}') \\
 &= b_s^\dagger(\mathbf{p}),
 \end{aligned}$$

$$\text{i.e. } b_s^\dagger(\mathbf{p}') = \int d^3\mathbf{x}' \tilde{\psi}_{p'\beta}^{(s)\dagger}(x') \psi_\beta(x'),$$

and therefore

$$b_r(\mathbf{p}) = \langle \psi, \tilde{\psi}_p^{(r)} \rangle = \int d^3\mathbf{x} \psi_\alpha^\dagger(x) \tilde{\psi}_{p\alpha}^{(r)}(x).$$

So

$$\begin{aligned}
\{b_r(\mathbf{p}), b_s^\dagger(\mathbf{p}')\} &= \int d^3\mathbf{x} d^3\mathbf{x}' \tilde{\psi}_{p'\beta}^{(s)\dagger}(x') \tilde{\psi}_{p\alpha}^{(r)}(x) \{\psi_\beta(x'), \psi_\alpha^\dagger(x)\} \\
&= \int d^3\mathbf{x} d^3\mathbf{x}' \tilde{\psi}_{p'\beta}^{(s)\dagger}(x') \tilde{\psi}_{p\alpha}^{(r)}(x) \delta_{\alpha\beta} \delta(\mathbf{x} - \mathbf{x}') \\
&= \int d^3\mathbf{x} \tilde{\psi}_{p'\alpha}^{(s)\dagger}(x) \tilde{\psi}_{p\alpha}^{(r)}(x) \\
&= \langle \tilde{\psi}_{p'}^{(s)}, \tilde{\psi}_p^{(r)} \rangle \\
&= \delta_{rs} 2p^0 (2\pi)^3 \delta(\mathbf{p} - \mathbf{p}').
\end{aligned}$$

(b) I'll give here the correct derivation, which requires a slight (but crucial) modification in the definition of P^μ .

$$\begin{aligned}
P^\mu &= \frac{i}{2} \int d^3\mathbf{x} \bar{\psi} \gamma^\mu \partial^0 \psi \\
&= \frac{i}{2} \frac{1}{(2\pi)^3} \sum_{r=1}^2 \int d^3\mathbf{x} \int \frac{d^3\mathbf{p}}{2p^0} \{ \bar{u}_r(p) e^{ip \cdot x} a_r^\dagger(\mathbf{p}) + \bar{v}_r(p) e^{-ip \cdot x} b_r(\mathbf{p}) \} \gamma^\mu \\
&\quad \frac{1}{(2\pi)^3} \sum_{s=1}^2 \int \frac{d^3\mathbf{p}'}{2p'^0} \{ (-ip'^0 u_s(p') e^{-ip' \cdot x} a_s(\mathbf{p}') + ip'^0 v_s(p') e^{ip' \cdot x} b_s^\dagger(\mathbf{p}') \} \\
&\quad - \frac{i}{2} \frac{1}{(2\pi)^3} \sum_{r=1}^2 \int d^3\mathbf{x} \int \frac{d^3\mathbf{p}}{2p^0} \{ ip^0 \bar{u}_r(p) e^{ip \cdot x} a_r^\dagger(\mathbf{p}) + (-ip^0) \bar{v}_r(p) e^{-ip \cdot x} b_r(\mathbf{p}) \} \gamma^\mu \\
&\quad \frac{1}{(2\pi)^3} \sum_{s=1}^2 \int \frac{d^3\mathbf{p}'}{2p'^0} \{ u_s(p') e^{-ip' \cdot x} a_s(\mathbf{p}') + v_s(p') e^{ip' \cdot x} b_s^\dagger(\mathbf{p}') \} \\
&= \frac{1}{2} \frac{1}{(2\pi)^6} \sum_{r=1}^2 \sum_{s=1}^2 \int d^3\mathbf{x} \int \frac{d^3\mathbf{p}}{2p^0} \int \frac{d^3\mathbf{p}'}{2p'^0} \\
&\quad \left[(p^0 + p'^0) e^{i(p-p') \cdot x} \bar{u}_r(p) \gamma^\mu u_s(p') a_r^\dagger(\mathbf{p}) a_s(\mathbf{p}') \right. \\
&\quad + (p^0 - p'^0) e^{i(p+p') \cdot x} \bar{u}_r(p) \gamma^\mu v_s(p') a_r^\dagger(\mathbf{p}) b_s^\dagger(\mathbf{p}') \\
&\quad + (-p^0 + p'^0) e^{i(-p-p') \cdot x} \bar{v}_r(p) \gamma^\mu u_s(p') b_r(\mathbf{p}) a_s(\mathbf{p}') \\
&\quad \left. + (-p^0 - p'^0) e^{i(p'-p) \cdot x} \bar{v}_r(p) \gamma^\mu v_s(p') b_r(\mathbf{p}) b_s^\dagger(\mathbf{p}') \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \frac{1}{(2\pi)^6} \sum_{r=1}^2 \sum_{s=1}^2 \int \frac{d^3 \mathbf{p}}{2p^0} \int \frac{d^3 \mathbf{p}'}{2p'^0} \\
&\quad \left[(p^0 + p'^0) e^{i(p^0 - p'^0)x^0} (2\pi)^3 \delta(\mathbf{p} - \mathbf{p}') \bar{u}_r(p) \gamma^\mu u_s(p') a_r^\dagger(\mathbf{p}) a_s(\mathbf{p}') \right. \\
&\quad + (p^0 - p'^0) e^{i(p^0 + p'^0)x^0} (2\pi)^3 \delta(\mathbf{p} + \mathbf{p}') \bar{u}_r(p) \gamma^\mu v_s(p') a_r^\dagger(\mathbf{p}) b_s^\dagger(\mathbf{p}') \\
&\quad + (-p^0 + p'^0) e^{i(-p^0 - p'^0)x^0} (2\pi)^3 \delta(\mathbf{p} + \mathbf{p}') \bar{v}_r(p) \gamma^\mu u_s(p') b_r(\mathbf{p}) a_s(\mathbf{p}') \\
&\quad \left. + (-p^0 - p'^0) e^{i(p'^0 - p^0)x^0} (2\pi)^3 \delta(\mathbf{p} - \mathbf{p}') \bar{v}_r(p) \gamma^\mu v_s(p') b_r(\mathbf{p}) b_s^\dagger(\mathbf{p}') \right] \\
&= \frac{1}{2} \frac{1}{(2\pi)^3} \sum_{r=1}^2 \sum_{s=1}^2 \int \frac{d^3 \mathbf{p}}{2p^0} \\
&\quad \frac{1}{2p^0} [2p^0 \bar{u}_r(p) \gamma^\mu u_s(p) a_r^\dagger(\mathbf{p}) a_s(\mathbf{p}) - 2p^0 \bar{v}_r(p) \gamma^\mu v_s(p) b_r(\mathbf{p}) b_s^\dagger(\mathbf{p})] \\
&= \frac{1}{2} \frac{1}{(2\pi)^3} \sum_{r=1}^2 \sum_{s=1}^2 \int \frac{d^3 \mathbf{p}}{2p^0} \\
&\quad [2p^\mu \delta_{rs} a_r^\dagger(\mathbf{p}) a_s(\mathbf{p}) - 2p^\mu \delta_{rs} b_r(\mathbf{p}) b_s^\dagger(\mathbf{p})] \\
&= \frac{1}{(2\pi)^3} \sum_{r=1}^2 \int \frac{d^3 \mathbf{p}}{2p^0} p^\mu [a_r^\dagger(\mathbf{p}) a_r(\mathbf{p}) - b_r(\mathbf{p}) b_r^\dagger(\mathbf{p})]
\end{aligned}$$

Proof that $\bar{u}_r(\mathbf{p}) \gamma^\mu u_s(\mathbf{p}) = 2p^\mu \delta_{rs}$. ■

$$\begin{aligned}
\bar{u}_r(\mathbf{p}) \gamma^i u_s(\mathbf{p}) &= (p^0 + m) \left(\chi_r^\dagger \quad \chi_r^\dagger \frac{\sigma \cdot \mathbf{p}}{p^0 + m} \right) \gamma^0 \gamma^i \left(\frac{\chi_s}{\frac{\sigma \cdot \mathbf{p}}{p^0 + m} \chi_s} \right) \\
&= (p^0 + m) \left(\chi_r^\dagger \quad \chi_r^\dagger \frac{\sigma \cdot \mathbf{p}}{p^0 + m} \right) \begin{pmatrix} 0_2 & \sigma_i \\ \sigma_i & 0_2 \end{pmatrix} \begin{pmatrix} \chi_s \\ \frac{\sigma \cdot \mathbf{p}}{p^0 + m} \chi_s \end{pmatrix} \\
&= (p^0 + m) \chi_r^\dagger \left(\sigma^i \frac{\sigma \cdot \mathbf{p}}{p^0 + m} + \frac{\sigma \cdot \mathbf{p}}{p^0 + m} \sigma^i \right) \chi_s \\
&= p^j \chi_r^\dagger (\sigma^i \sigma^j + \sigma^j \sigma^i) \chi_s \\
&= p^j 2\delta_{ij} \chi_r^\dagger \chi_s \\
&= 2p^i \delta_{rs}.
\end{aligned}$$

$$\begin{aligned}
\bar{u}_r(\mathbf{p})\gamma^0 u_s(\mathbf{p}) &= (p^0 + m) \begin{pmatrix} \chi_r^\dagger & \chi_r^\dagger \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{p^0 + m} \end{pmatrix} (\gamma^0)^2 \begin{pmatrix} \chi_s \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{p^0 + m} \chi_s \end{pmatrix} \\
&= (p^0 + m) \begin{pmatrix} \chi_r^\dagger & \chi_r^\dagger \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{p^0 + m} \end{pmatrix} \begin{pmatrix} \chi_s \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{p^0 + m} \chi_s \end{pmatrix} \\
&= (p^0 + m) \chi_r^\dagger \left[1 + \left(\frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{p^0 + m} \right)^2 \right] \chi_s \\
&= (p^0 + m) \chi_r^\dagger \left[1 + \frac{\mathbf{p}^2}{(p^0 + m)^2} \right] \chi_s \\
&= \left[p^0 + m + \frac{(p^0)^2 - m^2}{p^0 + m} \right] \delta_{rs} \\
&= (p^0 + m + p^0 - m) \delta_{rs} = 2p^0 \delta_{rs}
\end{aligned}$$

So $\bar{u}_r(\mathbf{p})\gamma^\mu u_s(\mathbf{p}) = 2p^\mu \delta_{rs}$