MATH425 Quantum Field Theory Solutions 8

1.

$$\begin{aligned}
\alpha_H =& U^\dagger \alpha U \\
i\hbar\frac{\partial \alpha_H}{\partial t} =& i\hbar \frac{\partial U^\dagger}{\partial t}\alpha U + i\hbar U^\dagger \alpha \frac{\partial U}{\partial t} \\
=& -\left(i\hbar\frac{\partial U}{\partial t}\right)^\dagger \alpha U + U^\dagger \alpha \left(i\hbar \frac{\partial U}{\partial t}\right) \\
=& -(HU)^\dagger \alpha U + U^\dagger \alpha H U \\
=& -U^\dagger H \alpha U + U^\dagger \alpha H U \quad (H^\dagger = H) \\
=& U^\dagger \alpha U U^\dagger H U - U^\dagger H U U^\dagger \alpha U \\
=& \alpha_H H_H - H_H \alpha_H = [\alpha_H, H_H].
\end{aligned}$$

2. Two-particle states are defined by

$$\begin{aligned}
|\mathbf{p}_1, \mathbf{p}_2 >=& a^\dagger(\mathbf{p}_1) a^\dagger(\mathbf{p}_1)|0> \\
|\mathbf{p}_1, \mathbf{p}_2 >=& |\mathbf{p}_2, \mathbf{p}_1 > \quad \text{as} \quad [a(\mathbf{p}_1), a(\mathbf{p}_2)] = 0 \\
< \mathbf{p}_1', \mathbf{p}_2' | \mathbf{p}_1, \mathbf{p}_2 >=& < 0|a(\mathbf{p}_1')a(\mathbf{p}_2')a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)|0> \\
=& < 0|a(\mathbf{p}_1')\{a^\dagger(\mathbf{p}_1)a(\mathbf{p}_2') + (2\pi)^3 2p_1^0 \delta(\mathbf{p}_1 - \mathbf{p}_2')\}a^\dagger(\mathbf{p}_2)|0> \\
=& < 0|a(\mathbf{p}_1')a^\dagger(\mathbf{p}_1)\{a^\dagger(\mathbf{p}_2)a(\mathbf{p}_2') + (2\pi)^3 2p_1^0 \delta(\mathbf{p}_2 - \mathbf{p}_2')\}|0> \\
& + (2\pi)^3 2p_1^0 \delta(\mathbf{p}_1 - \mathbf{p}_2') < 0|a(\mathbf{p}_1')a^\dagger(\mathbf{p}_2)|0> \\
=& (2\pi)^3 2p_2^0 \delta(\mathbf{p}_2 - \mathbf{p}_2') < 0|\{a^\dagger(\mathbf{p}_1)a(\mathbf{p}_1') + (2\pi)^3 2p_1^0 \delta(\mathbf{p}_1 - \mathbf{p}_1')\}|0> \\
& + (2\pi)^3 2p_1^0 \delta(\mathbf{p}_1 - \mathbf{p}_2') < 0|\{a^\dagger(\mathbf{p}_2)a(\mathbf{p}_1') + (2\pi)^3 2p_2^0 \delta(\mathbf{p}_2 - \mathbf{p}_1')\}|0> \\
=& (2\pi)^6 (2p_1^0)(2p_2^0)\{\delta(\mathbf{p}_1 - \mathbf{p}_1')\delta(\mathbf{p}_2 - \mathbf{p}_2') + \delta(\mathbf{p}_1 - \mathbf{p}_2')\delta(\mathbf{p}_2 - \mathbf{p}_1')\}.
\end{aligned}$$

3.

$$\begin{aligned}
\int |\tilde{\psi}(\mathbf{p})|^2 d^3\mathbf{p} =& \frac{1}{(2\pi)^3}\int d^3\mathbf{p} d^3\mathbf{x} d^3\mathbf{x}' \psi^*(\mathbf{x})\psi(\mathbf{x}') e^{-i\mathbf{p}.\mathbf{x}} e^{i\mathbf{p}.\mathbf{x}'} \\
=& \frac{1}{(2\pi)^3}\int d^3\mathbf{p} d^3\mathbf{x} d^3\mathbf{x}' \psi^*(\mathbf{x})\psi(\mathbf{x}') e^{i\mathbf{p}.(\mathbf{x}' - \mathbf{x})} \\
=& \frac{1}{(2\pi)^3}\int d^3\mathbf{x} d^3\mathbf{x}' \psi^*(\mathbf{x})\psi(\mathbf{x}')(2\pi)^3\delta(\mathbf{x} - \mathbf{x}') \\
=& \int d^3\mathbf{x} \psi^*(\mathbf{x})\psi(\mathbf{x}) = \int d^3\mathbf{x} |\psi(\mathbf{x})|^2 = 1.
\end{aligned}$$

4.

$$
\begin{aligned}
N =& \frac{1}{(2\pi)^3}\int \frac{d^3\mathbf{p}'}{2p'^0} a^\dagger(\mathbf{p}')a(\mathbf{p}') \\
\Rightarrow [N, a^\dagger(\mathbf{p})] =& \left[\frac{1}{(2\pi)^3}\int \frac{d^3\mathbf{p}'}{2p'^0} a^\dagger(\mathbf{p}')a(\mathbf{p}'), a^\dagger(\mathbf{p})\right] \\
=& \frac{1}{(2\pi)^3}\int \frac{d^3\mathbf{p}'}{2p'^0} [a^\dagger(\mathbf{p}')a(\mathbf{p}'), a^\dagger(\mathbf{p})] \\
=& \frac{1}{(2\pi)^3}\int \frac{d^3\mathbf{p}'}{2p'^0} \left\{a^\dagger(\mathbf{p}')[a(\mathbf{p}'), a^\dagger(\mathbf{p})] + [a^\dagger(\mathbf{p}'), a^\dagger(\mathbf{p})]a(\mathbf{p}')\right\} \\
=& \frac{1}{(2\pi)^3}\int \frac{d^3\mathbf{p}'}{2p'^0} a^\dagger(\mathbf{p}') 2p'^0 (2\pi)^3 \delta(\mathbf{p}-\mathbf{p}') \\
=& a^\dagger(\mathbf{p}).
\end{aligned}
$$

So we have

$$
\begin{aligned}
Na^\dagger(\mathbf{p}) - a^\dagger(\mathbf{p})N =& a^\dagger(\mathbf{p}) \\
\Rightarrow Na^\dagger(\mathbf{p}) =& a^\dagger(\mathbf{p})(N+1) \\
\Rightarrow N|\mathbf{p}_1 \ldots \mathbf{p}_n> =& Na^\dagger(\mathbf{p}_1)\ldots a^\dagger(\mathbf{p}_n)|0> \\
=& a^\dagger(\mathbf{p}_1)(N+1)a^\dagger(\mathbf{p}_2)\ldots a^\dagger(\mathbf{p}_n)|0> \\
=& a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)(N+2)\ldots a^\dagger(\mathbf{p}_n)|0> \\
=& \ldots = a^\dagger(\mathbf{p}_1)\ldots a^\dagger(\mathbf{p}_n)(N+n)|0> \\
=& na^\dagger(\mathbf{p}_1)\ldots a^\dagger(\mathbf{p}_n)|0> \quad (a(\mathbf{p})|0>=0) \\
=& n|\mathbf{p}_1 \ldots \mathbf{p}_n> .
\end{aligned}
$$