

MATH425 Quantum Field Theory Solutions 7

1(a).

$$\begin{aligned}
 u_r(p) &= \sqrt{p^0 + m} \begin{pmatrix} \chi_r \\ \frac{\sigma \cdot \mathbf{p}}{p^0 + m} \chi_r \end{pmatrix} \\
 \Rightarrow \overline{u_r(p)} u_s(p) &= u_r^\dagger \gamma^0 u_s = (p^0 + m) \begin{pmatrix} \chi_r^\dagger & \chi_r^\dagger \frac{\sigma \cdot \mathbf{p}}{p^0 + m} \end{pmatrix} \begin{pmatrix} 1_2 & 0_2 \\ 0_2 & -1_2 \end{pmatrix} \begin{pmatrix} \chi_s \\ \frac{\sigma \cdot \mathbf{p}}{p^0 + m} \chi_s \end{pmatrix} \\
 &= (p^0 + m) \left(\chi_r^\dagger \chi_s - \chi_r^\dagger \frac{(\sigma \cdot \mathbf{p})^2}{(p^0 + m)^2} \chi_s \right) \\
 &= (p^0 + m) \left(\chi_r^\dagger \chi_s - \chi_r^\dagger \frac{\mathbf{p}^2}{(p^0 + m)^2} \chi_s \right) \\
 &= (p^0 + m) \left(1 - \frac{\mathbf{p}^2}{(p^0 + m)^2} \right) \chi_r^\dagger \chi_s \\
 &= (p^0 + m) \left(\frac{(p^0 + m)^2 - ((p^0)^2 - m^2)}{(p^0 + m)^2} \right) \delta_{rs} \\
 &= (p^0 + m) \frac{2p^0 m + 2m^2}{(p^0 + m)^2} \delta_{rs} = 2m \delta_{rs}.
 \end{aligned}$$

(b).

$$\begin{aligned}
 u_1(p) \overline{u_1(p)} &= u_1(p) u_1^\dagger \gamma^0 = (p^0 + m) \begin{pmatrix} \chi_1 \\ \frac{\sigma \cdot \mathbf{p}}{p^0 + m} \chi_1 \end{pmatrix} \begin{pmatrix} \chi_1^\dagger & \chi_1^\dagger \frac{\sigma \cdot \mathbf{p}}{p^0 + m} \end{pmatrix} \begin{pmatrix} 1_2 & 0_2 \\ 0_2 & -1_2 \end{pmatrix} \\
 &= (p^0 + m) \begin{pmatrix} \chi_1 \\ \frac{\sigma \cdot \mathbf{p}}{p^0 + m} \chi_1 \end{pmatrix} \begin{pmatrix} \chi_1^\dagger & -\chi_1^\dagger \frac{\sigma \cdot \mathbf{p}}{p^0 + m} \end{pmatrix} \\
 &= (p^0 + m) \begin{pmatrix} \chi_1 \chi_1^\dagger & -\chi_1 \chi_1^\dagger \frac{\sigma \cdot \mathbf{p}}{p^0 + m} \\ \frac{\sigma \cdot \mathbf{p}}{p^0 + m} \chi_1 \chi_1^\dagger & -\frac{\sigma \cdot \mathbf{p}}{p^0 + m} \chi_1 \chi_1^\dagger \frac{\sigma \cdot \mathbf{p}}{p^0 + m} \end{pmatrix}
 \end{aligned}$$

So

$$\begin{aligned}
& u_1(p)\overline{u_1(p)} + u_2(p)\overline{u_2(p)} \\
&= (p^0 + m) \begin{pmatrix} (\chi_1\chi_1^\dagger + \chi_2\chi_2^\dagger) & -(\chi_1\chi_1^\dagger + \chi_2\chi_2^\dagger)\frac{\sigma\cdot\mathbf{p}}{p^0+m} \\ \frac{\sigma\cdot\mathbf{p}}{p^0+m}(\chi_1\chi_1^\dagger + \chi_2\chi_2^\dagger) & -\frac{\sigma\cdot\mathbf{p}}{p^0+m}(\chi_1\chi_1^\dagger + \chi_2\chi_2^\dagger)\frac{\sigma\cdot\mathbf{p}}{p^0+m} \end{pmatrix} \\
&= (p^0 + m) \begin{pmatrix} 1 & -\frac{\sigma\cdot\mathbf{p}}{p^0+m} \\ \frac{\sigma\cdot\mathbf{p}}{p^0+m} & -\frac{(\sigma\cdot\mathbf{p})^2}{(p^0+m)^2} \end{pmatrix} \\
&= (p^0 + m) \begin{pmatrix} 1 & -\frac{\sigma\cdot\mathbf{p}}{p^0+m} \\ \frac{\sigma\cdot\mathbf{p}}{p^0+m} & -\frac{(\mathbf{p})^2}{(p^0+m)^2} \end{pmatrix} \\
&= (p^0 + m) \begin{pmatrix} 1 & -\frac{\sigma\cdot\mathbf{p}}{p^0+m} \\ \frac{\sigma\cdot\mathbf{p}}{p^0+m} & -\frac{(p^0)^2 - m^2}{(p^0+m)^2} \end{pmatrix} \\
&= (p^0 + m) \begin{pmatrix} 1 & -\frac{\sigma\cdot\mathbf{p}}{p^0+m} \\ \frac{\sigma\cdot\mathbf{p}}{p^0+m} & -\frac{p^0 - m}{p^0+m} \end{pmatrix} \\
&= \begin{pmatrix} p^0 + m & -\sigma\cdot\mathbf{p} \\ \sigma\cdot\mathbf{p} & -p^0 + m \end{pmatrix} \\
&= \eta_{\mu\nu}\gamma^\mu p^\nu + m = \gamma\cdot p + m.
\end{aligned}$$

So $P_+ = \frac{1}{2m}(u_1(p)\overline{u_1(p)} + u_2(p)\overline{u_2(p)})$

$$= \frac{1}{2m}(\gamma\cdot p + m).$$

(We have used $\chi_1\chi_1^\dagger + \chi_2\chi_2^\dagger = 1_2$.) So

$$\begin{aligned}
P_+^2 &= \frac{1}{4m^2}(u_1(p)\overline{u_1(p)} + u_2(p)\overline{u_2(p)})(u_1(p)\overline{u_1(p)} + u_2(p)\overline{u_2(p)}) \\
&= \frac{1}{4m^2}[(u_1(p)\overline{u_1(p)}u_1(p)\overline{u_1(p)} + u_1(p)\overline{u_1(p)}u_2(p)\overline{u_2(p)} \\
&\quad + u_2(p)\overline{u_2(p)}u_1(p)\overline{u_1(p)} + u_2(p)\overline{u_2(p)}u_2(p)\overline{u_2(p)})] \\
&= \frac{1}{4m^2}(u_1(p)2m\overline{u_1(p)} + 0 + 0 + u_2(p)2m\overline{u_2(p)}) \\
&= \frac{1}{2m}(u_1(p)\overline{u_1(p)} + u_2(p)\overline{u_2(p)}) = P_+.
\end{aligned}$$

Or

$$\begin{aligned}
P_+^2 &= \frac{1}{4m^2}(\gamma\cdot p + m)^2 = \frac{1}{4m^2}[(\gamma\cdot p)^2 + 2m\gamma\cdot p + m^2] \\
&= \frac{1}{4m^2}[p^2 + 2m\gamma\cdot p + m^2] = \frac{1}{4m^2}[m^2 + 2m\gamma\cdot p + m^2] \\
&= \frac{1}{4m^2}[2m^2 + 2m\gamma\cdot p] = \frac{1}{2m}[\gamma\cdot p + m] = P_+.
\end{aligned}$$

Similarly,

$$\begin{aligned}
v_1(p)\overline{v_1(p)} &= v_1(p)v_1^\dagger\gamma^0 = (p^0 + m) \begin{pmatrix} \frac{\sigma \cdot \mathbf{p}}{p^0 + m} \chi_1 \\ \chi_1 \end{pmatrix} \begin{pmatrix} \chi_1^\dagger \frac{\sigma \cdot \mathbf{p}}{p^0 + m} & \chi_1^\dagger \end{pmatrix} \begin{pmatrix} 1_2 & 0_2 \\ 0_2 & -1_2 \end{pmatrix} \\
&= (p^0 + m) \begin{pmatrix} \frac{\sigma \cdot \mathbf{p}}{p^0 + m} \chi_1 \\ \chi_1 \end{pmatrix} \begin{pmatrix} \chi_1^\dagger \frac{\sigma \cdot \mathbf{p}}{p^0 + m} & -\chi_1^\dagger \end{pmatrix} \\
&= (p^0 + m) \begin{pmatrix} \frac{\sigma \cdot \mathbf{p}}{p^0 + m} \chi_1 \chi_1^\dagger \frac{\sigma \cdot \mathbf{p}}{p^0 + m} & -\frac{\sigma \cdot \mathbf{p}}{p^0 + m} \chi_1 \chi_1^\dagger \\ \chi_1 \chi_1^\dagger \frac{\sigma \cdot \mathbf{p}}{p^0 + m} & -\chi_1 \chi_1^\dagger \end{pmatrix}
\end{aligned}$$

So

$$\begin{aligned}
&v_1(p)\overline{v_1(p)} + v_2(p)\overline{v_2(p)} \\
&= (p^0 + m) \begin{pmatrix} \frac{\sigma \cdot \mathbf{p}}{p^0 + m} (\chi_1 \chi_1^\dagger + \chi_2 \chi_2^\dagger) \frac{\sigma \cdot \mathbf{p}}{p^0 + m} & -\frac{\sigma \cdot \mathbf{p}}{p^0 + m} (\chi_1 \chi_1^\dagger + \chi_2 \chi_2^\dagger) \\ (\chi_1 \chi_1^\dagger + \chi_2 \chi_2^\dagger) \frac{\sigma \cdot \mathbf{p}}{p^0 + m} & -(\chi_1 \chi_1^\dagger + \chi_2 \chi_2^\dagger) \end{pmatrix} \\
&= (p^0 + m) \begin{pmatrix} \frac{(\sigma \cdot \mathbf{p})^2}{(p^0 + m)^2} & -\frac{\sigma \cdot \mathbf{p}}{p^0 + m} \\ \frac{\sigma \cdot \mathbf{p}}{p^0 + m} & -1 \end{pmatrix} \\
&= (p^0 + m) \begin{pmatrix} \frac{(\mathbf{p})^2}{(p^0 + m)^2} & -\frac{\sigma \cdot \mathbf{p}}{p^0 + m} \\ \frac{\sigma \cdot \mathbf{p}}{p^0 + m} & -1 \end{pmatrix} \\
&= (p^0 + m) \begin{pmatrix} \frac{(p^0)^2 - m^2}{(p^0 + m)^2} & -\frac{\sigma \cdot \mathbf{p}}{p^0 + m} \\ \frac{\sigma \cdot \mathbf{p}}{p^0 + m} & -1 \end{pmatrix} \\
&= (p^0 + m) \begin{pmatrix} \frac{p^0 - m}{p^0 + m} & -\frac{\sigma \cdot \mathbf{p}}{p^0 + m} \\ \frac{\sigma \cdot \mathbf{p}}{p^0 + m} & -1 \end{pmatrix} \\
&= \begin{pmatrix} p^0 - m & -\sigma \cdot \mathbf{p} \\ \sigma \cdot \mathbf{p} & -p^0 - m \end{pmatrix} \\
&= \gamma \cdot p - m.
\end{aligned}$$

$$\begin{aligned}
\text{So } P_- &= -\frac{1}{2m} (v_1(p)\overline{v_1(p)} + v_2(p)\overline{v_2(p)}) \\
&= -\frac{1}{2m} (-\gamma \cdot p + m).
\end{aligned}$$

So (this wasn't asked for in the question but we'll do it anyway)

$$\begin{aligned}
P_-^2 &= \frac{1}{4m^2} (v_1(p)\overline{v_1(p)} + v_2(p)\overline{v_2(p)}) (v_1(p)\overline{v_1(p)} + v_2(p)\overline{v_2(p)}) \\
&= \frac{1}{4m^2} [v_1(p)\overline{v_1(p)}v_1(p)\overline{v_1(p)} + v_1(p)\overline{v_1(p)}v_2(p)\overline{v_2(p)} \\
&\quad + v_2(p)\overline{v_2(p)}v_1(p)\overline{v_1(p)} + v_2(p)\overline{v_2(p)}v_2(p)\overline{v_2(p)}] \\
&= \frac{1}{4m^2} (v_1(p)(-2m)\overline{v_1(p)} + 0 + 0 + v_2(p)(-2m)\overline{v_2(p)}) \\
&= -\frac{1}{2m} (v_1(p)\overline{v_1(p)} + v_2(p)\overline{v_2(p)}) = P_-.
\end{aligned}$$

Or

$$\begin{aligned} P_-^2 &= \frac{1}{4m^2} (-\gamma \cdot p + m)^2 = \frac{1}{4m^2} [(\gamma \cdot p)^2 - 2m\gamma \cdot p + m^2] \\ &= \frac{1}{4m^2} [p^2 - 2m\gamma \cdot p + m^2] = \frac{1}{4m^2} [m^2 - 2m\gamma \cdot p + m^2] \\ &= \frac{1}{4m^2} [2m^2 - 2m\gamma \cdot p] = \frac{1}{2m} [-\gamma \cdot p + m] = P_- . \end{aligned}$$