MATH425 Quantum Field Theory Solutions 4

1. With $\rho = |\psi|^2 = \psi^*\psi$, we have

$$\frac{\partial \rho}{\partial t} = \frac{\partial \psi^*}{\partial t}\psi + \psi^* \frac{\partial \psi}{\partial t}.$$

Now from the Schrödinger equation,

$$i\hbar \frac{\partial \psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x}) \right) \psi, \tag{1}$$

and so, taking the complex conjugate

$$-i\hbar \frac{\partial \psi^*}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x}) \right) \psi^* \tag{2}$$

(assuming that $V(\mathbf{x})$ is real.) Multiplying (1) by ψ^* and (2) by ψ and subtracting, we have

$$i\hbar \left(\psi^* \frac{\partial \psi}{\partial t} + \frac{\partial \psi^*}{\partial t}\psi \right) = -\frac{\hbar^2}{2m}\left(\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^* \right) = -\frac{\hbar^2}{2m}\nabla . \left[\psi^* \nabla \psi - \psi \nabla \psi^* \right],$$

and so

$$\frac{\partial \rho}{\partial t} = \frac{i\hbar}{2m}\nabla . \left[\psi^* \nabla \psi - \psi \nabla \psi^* \right],$$

i.e.

$$\frac{\partial \rho}{\partial t} + \text{div}\mathbf{j} = 0,$$

$$\text{where} \quad \mathbf{j} = -\frac{i\hbar}{2m}\left[\psi^* \nabla \psi - \psi \nabla \psi^* \right]$$

2.

$$\phi = \frac{1}{(2\pi)^3}\int d^3\mathbf{p} \frac{1}{2p^0}\left[f_+(\mathbf{p})e^{-ip.x} + f_-(\mathbf{p})e^{ip.x} \right]$$

where $p^0 = \sqrt{\mathbf{p}^2 + m^2}$. So

$$
\begin{aligned}
||\phi||^2 =& i \int \phi(x)^* \partial^0 \phi(x) d^3\mathbf{x} \\
=& \frac{i}{(2\pi)^6} \int d^3\mathbf{x} \int d^3\mathbf{p}' \frac{1}{2p'^0} \left[f_+^*(\mathbf{p}') e^{ip'.x} + f_-^*(\mathbf{p}') e^{-ip'.x} \right] \partial^0 \\
& \int d^3\mathbf{p} \frac{1}{2p^0} \left[f_+(\mathbf{p}) e^{-ip.x} + f_-(\mathbf{p}) e^{ip.x} \right] \\
=& \frac{1}{(2\pi)^6} \int d^3\mathbf{x} \int d^3\mathbf{p}' \frac{1}{2p'^0} \int d^3\mathbf{p} \frac{1}{2p^0} \Big[(p'^0 + p^0) f_+^*(\mathbf{p}') f_+(\mathbf{p}) e^{i(p'-p).x} \\
& + (p'^0 - p^0) f_+^*(\mathbf{p}') f_-(\mathbf{p}) e^{i(p'+p).x} + (-p'^0 + p^0) f_-^*(\mathbf{p}') f_+(\mathbf{p}) e^{-i(p'+p).x} \\
& + (-p'^0 - p^0) f_-^*(\mathbf{p}') f_-(\mathbf{p}) e^{i(p-p').x} \Big] \\
=& \frac{1}{(2\pi)^6} \int d^3\mathbf{x} \int d^3\mathbf{p}' \frac{1}{2p'^0} \int d^3\mathbf{p} \frac{1}{2p^0} \\
& \Big[(p'^0 + p^0) f_+^*(\mathbf{p}') f_+(\mathbf{p}) e^{i(p'^0 - p^0)x^0} e^{i(\mathbf{p}-\mathbf{p}').\mathbf{x}} \\
& + (p'^0 - p^0) f_+^*(\mathbf{p}') f_-(\mathbf{p}) e^{i(p'^0 + p^0)x^0} e^{-i(\mathbf{p}'+\mathbf{p}).\mathbf{x}} \\
& + (-p'^0 + p^0) f_-^*(\mathbf{p}') f_+(\mathbf{p}) e^{-i(p'^0 + p^0)x^0} e^{i(\mathbf{p}'+\mathbf{p}).\mathbf{x}} \\
& + (-p'^0 - p^0) f_-^*(\mathbf{p}') f_-(\mathbf{p}) e^{i(p^0 - p'^0)x^0} e^{i(\mathbf{p}'-\mathbf{p}).\mathbf{x}} \Big] \\
=& \frac{1}{(2\pi)^3} \int d^3\mathbf{x} \int d^3\mathbf{p}' \frac{1}{2p'^0} \int d^3\mathbf{p} \frac{1}{2p^0} \\
& \Big[(p'^0 + p^0) f_+^*(\mathbf{p}') f_+(\mathbf{p}) e^{i(p'^0 - p^0)x^0} \delta(\mathbf{p}-\mathbf{p}') \\
& + (p'^0 - p^0) f_+^*(\mathbf{p}') f_-(\mathbf{p}) e^{i(p'^0 + p^0)x^0} \delta(\mathbf{p}+\mathbf{p}') \\
& + (-p'^0 + p^0) f_-^*(\mathbf{p}') f_+(\mathbf{p}) e^{-i(p'^0 + p^0)x^0} \delta(\mathbf{p}+\mathbf{p}') \\
& + (-p'^0 - p^0) f_-^*(\mathbf{p}') f_-(\mathbf{p}) e^{i(p^0 - p'^0)x^0} \delta(\mathbf{p}-\mathbf{p}') \Big],
\end{aligned}
$$

where we have used $\int d^3\mathbf{x} e^{i\mathbf{p}.\mathbf{x}} = (2\pi)^3 \delta(\mathbf{p})$ and $\delta(-x) = \delta(x)$. Doing the integral over $\mathbf{p}'$, $\delta(\mathbf{p}-\mathbf{p}')$ sets $\mathbf{p}' = \mathbf{p}$ and $\delta(\mathbf{p}+\mathbf{p}')$ sets $\mathbf{p}' = -\mathbf{p}$. Recalling that in these expressions we have $p^0 = \sqrt{\mathbf{p}^2 + m^2}$, $p'^0 = \sqrt{\mathbf{p}'^2 + m^2}$, we have $p'^0 = p^0$ and hence

$$
||\phi||^2 = \frac{1}{(2\pi)^3} \int d^3\mathbf{p} \frac{1}{2p^0} \left[|f_+(\mathbf{p})|^2 - |f_-(\mathbf{p})|^2 \right],
$$