MATH425 Quantum Field Theory Solutions 11

1.

$$
\begin{aligned}
[P^\mu, \phi(x)] =&\Big[\frac{1}{(2\pi)^3}\int d^3\mathbf{p}\frac{1}{2p^0}p^\mu a^\dagger(\mathbf{p})a(\mathbf{p}),\\
&\frac{1}{(2\pi)^3}\int d^3\mathbf{p}'\frac{1}{2p'^0}\left\{a(\mathbf{p}')e^{-ip'.x}+a^\dagger(\mathbf{p}')e^{ip'.x}\right\}\Big]\\
=&\frac{1}{(2\pi)^6}\int d^3\mathbf{p}d^3\mathbf{p}'\frac{1}{2p^0}\frac{1}{2p'^0}p^\mu\{[a^\dagger(\mathbf{p})a(\mathbf{p}),a(\mathbf{p}')]e^{-ip'.x}\\
&+[a^\dagger(\mathbf{p})a(\mathbf{p}),a^\dagger(\mathbf{p}')]e^{ip'.x}\}.
\end{aligned}
$$

Now

$$
\begin{aligned}
[a^\dagger(\mathbf{p})a(\mathbf{p}),a(\mathbf{p}')] =&a^\dagger(\mathbf{p})[a(\mathbf{p}),a(\mathbf{p}')]+[a^\dagger(\mathbf{p}),a(\mathbf{p}')]a(\mathbf{p})\\
=&-(2\pi)^3 2p'^0\delta(\mathbf{p}-\mathbf{p}')a(\mathbf{p}),\\
[a^\dagger(\mathbf{p})a(\mathbf{p}),a^\dagger(\mathbf{p}')] =&a^\dagger(\mathbf{p})[a(\mathbf{p}),a^\dagger(\mathbf{p}')]+[a^\dagger(\mathbf{p}),a^\dagger(\mathbf{p}')]a(\mathbf{p})\\
=\ a^\dagger(\mathbf{p})(2\pi)^3 2p'^0\delta(\mathbf{p}-\mathbf{p}').
\end{aligned}
$$

So

$$
\begin{aligned}
[P^\mu,\phi(x)] =&\frac{1}{(2\pi)^3}\int d^3\mathbf{p}\frac{1}{2p^0}p^\mu[-a(\mathbf{p})e^{-ip.x}+a^\dagger(\mathbf{p})e^{ip.x}]\\
=&-i\partial^\mu\phi(x).
\end{aligned}
$$

So

$$
\begin{aligned}
e^{-ia.P}\phi(x)e^{ia.P} =&(1-ia.P)\phi(x)(1+ia.P)+O(a^2)\\
=&\phi(x)+i\phi(x)a.P-ia.P\phi(x)+O(a^2)\\
=&\phi(x)+ia_\mu(\phi(x)P^\mu-P^\mu\phi(x))+O(a^2)\\
=&\ \phi(x)-ia_\mu[P^\mu,\phi]+O(a^2)\\
=&\phi(x)-a_\mu\partial^\mu\phi(x)+O(a^2)=\phi(x-a)
\end{aligned}
$$

using Taylor's Theorem to 1st order.

2.

$$\begin{aligned}
\phi(x) =& \frac{1}{(2\pi)^3}\int d^3\mathbf{p}\frac{1}{2p^0}\left[a(\mathbf{p})e^{-ip.x} + a^\dagger(\mathbf{p})e^{ip.x}\right] \\
\Rightarrow \pi(x) = \dot{\phi}(x) =& \frac{1}{(2\pi)^3}\int d^3\mathbf{p}\frac{1}{2p^0}\left[-ip^0a(\mathbf{p})e^{-ip.x} + ip^0a^\dagger(\mathbf{p})e^{ip.x}\right] \\
\Rightarrow \int [p'^0\phi(x) - i\pi(x)]e^{-ip'.x}d^3\mathbf{x} =& \frac{1}{(2\pi)^3}\int d^3\mathbf{p}[(p'^0 - p^0)a(\mathbf{p})e^{-i(p'^0+p^0)x^0}e^{i(\mathbf{p}+\mathbf{p}').\mathbf{x}} \\
& + (p'^0 + p^0)a^\dagger(\mathbf{p})e^{i(p^0-p'^0)x^0}e^{-i(\mathbf{p}-\mathbf{p}').\mathbf{x}}]d^3\mathbf{x} \\
=& \frac{1}{(2\pi)^3}\int d^3\mathbf{p}[(p'^0 - p^0)a(\mathbf{p})e^{-i(p'^0+p^0)x^0}(2\pi)^3\delta(\mathbf{p}+\mathbf{p}') \\
& + (p'^0 + p^0)a^\dagger(\mathbf{p})e^{i(p^0-p'^0)x^0}(2\pi)^3\delta(\mathbf{p}-\mathbf{p}')]d^3\mathbf{x} \\
=& a^\dagger(\mathbf{p}') \\
\text{and similarly} \quad & \int [p^0\phi(x) + i\pi(x)]e^{ip'.x}d^3\mathbf{x} = a(\mathbf{p}').
\end{aligned}$$

Then

$$\begin{aligned}
[a(\mathbf{p}), a^\dagger(\mathbf{p}')] &= \left[\int [p^0\phi(x) + i\pi(x)]e^{ip.x}d^3\mathbf{x}, \int [p'^0\phi(x') - i\pi(x')]e^{-ip'.x'}d^3\mathbf{x}'\right] \\
&= \int d^3\mathbf{x}d^3\mathbf{x}'\left\{-ip^0[\phi(x),\pi(x')] + ip'^0[\pi(x),\phi(x')]\right\}e^{ip.x}e^{-ip'.x'}
\end{aligned}$$

Now x^0 and x'^0 are arbitrary, so we can choose $x^0 = x'^0 = t$ say, in order to usde the equal time commutation relations. Then

$$\begin{aligned}
[a(\mathbf{p}), a^\dagger(\mathbf{p}')] =& \int d^3\mathbf{x}d^3\mathbf{x}'\left\{-ip^0[\phi(\mathbf{x},t),\pi(\mathbf{x}',t)] + ip'^0[\pi(\mathbf{x},t),\phi(\mathbf{x}',t)]\right\}e^{ip.x}e^{-ip'.x'} \\
=& \int d^3\mathbf{x}d^3\mathbf{x}'[p^0 + p'^0]\delta(\mathbf{x}-\mathbf{x}')e^{ip.x}e^{-ip'.x'} \\
=& \int d^3\mathbf{x}[p^0 + p'^0]e^{i(p^0-p'^0)t}e^{i(\mathbf{p}'-\mathbf{p}).\mathbf{x}} \\
=& (2\pi)^3\delta(\mathbf{p}-\mathbf{p}')[p^0 + p'^0]e^{i(p^0-p'^0)t} \\
=& (2\pi)^3 2p^0\delta(\mathbf{p}-\mathbf{p}')
\end{aligned}$$

Also

$$
\begin{aligned}
[a^\dagger(\mathbf{p}), a^\dagger(\mathbf{p}')] &= \left[\int [p^0\phi(x) - i\pi(x)]e^{-ip.x}d^3\mathbf{x}, \int [p'^0\phi(x') - i\pi(x')]e^{-ip'.x'}d^3\mathbf{x}'\right] \\
&= \int d^3\mathbf{x}d^3\mathbf{x}' \left\{ip^0[\phi(\mathbf{x},t), \pi(\mathbf{x}',t)] + ip'^0[\pi(\mathbf{x},t), \phi(\mathbf{x}',t)]\right\} e^{-ip.x}e^{-ip'.x'} \\
&= \int d^3\mathbf{x}d^3\mathbf{x}'[-p^0 + p'^0]\delta(\mathbf{x} - \mathbf{x}')e^{-ip.x}e^{-ip'.x'} \\
&= \int d^3\mathbf{x}[-p^0 + p'^0]e^{-i(p^0+p'^0)t}e^{i(\mathbf{p}'+\mathbf{p}).\mathbf{x}} \\
&= (2\pi)^3\delta(\mathbf{p} + \mathbf{p}')[-p^0 + p'^0]e^{-i(p^0+p'^0)t} \\
&= 0.
\end{aligned}
$$

3.

$$
\begin{aligned}
\phi^{(+)}(x_1)\phi^{(+)}(x_2)|\mathbf{p}_1\mathbf{p}_2> =& \frac{1}{(2\pi)^3}\int\frac{d^3\mathbf{p}}{2p^0}a(\mathbf{p})e^{-ip.x_1}\frac{1}{(2\pi)^3}\int\frac{d^3\mathbf{p}'}{2p'^0}a(\mathbf{p}')e^{-ip'.x_2} \\
& a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)|0> \\
=& \frac{1}{(2\pi)^6}\int\frac{d^3\mathbf{p}}{2p^0}\int\frac{d^3\mathbf{p}'}{2p'^0}e^{-i(p.x_1+p'.x_2)} \\
& a(\mathbf{p})a(\mathbf{p}')a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)|0> \\
=& \frac{1}{(2\pi)^6}\int\frac{d^3\mathbf{p}}{2p^0}\int\frac{d^3\mathbf{p}'}{2p'^0}e^{-i(p.x_1+p'.x_2)} \\
& a(\mathbf{p})\Big\{[a(\mathbf{p}'),a^\dagger(\mathbf{p}_1)]+a^\dagger(\mathbf{p}_1)a(\mathbf{p}')\Big\}a^\dagger(\mathbf{p}_2)|0> \\
=& \frac{1}{(2\pi)^6}\int\frac{d^3\mathbf{p}}{2p^0}\int\frac{d^3\mathbf{p}'}{2p'^0}e^{-i(p.x_1+p'.x_2)} \\
& \Big\{(2\pi)^3 2p_1^0\delta(\mathbf{p}'-\mathbf{p}_1)a(\mathbf{p}')a^\dagger(\mathbf{p}_2)+a(\mathbf{p})a^\dagger(\mathbf{p}_1)a(\mathbf{p}')a^\dagger(\mathbf{p}_2)\Big\}|0> \\
=& \frac{1}{(2\pi)^6}\int\frac{d^3\mathbf{p}}{2p^0}\int\frac{d^3\mathbf{p}'}{2p'^0}e^{-i(p.x_1+p'.x_2)} \\
& \Big\{(2\pi)^3 2p_1^0\delta(\mathbf{p}'-\mathbf{p}_1)\Big([a(\mathbf{p}),a^\dagger(\mathbf{p}_2)]+a^\dagger(\mathbf{p}_2)a(\mathbf{p})\Big) \\
& +a(\mathbf{p})a^\dagger(\mathbf{p}_1)\Big([a(\mathbf{p}'),a^\dagger(\mathbf{p}_2)]+a^\dagger(\mathbf{p}_2)a(\mathbf{p}')\Big)\Big\}|0> \\
=& \frac{1}{(2\pi)^6}\int\frac{d^3\mathbf{p}}{2p^0}\int\frac{d^3\mathbf{p}'}{2p'^0}e^{-i(p.x_1+p'.x_2)} \\
& \Big\{(2\pi)^3 2p_1^0\delta(\mathbf{p}'-\mathbf{p}_1)(2\pi)^3 2p_2^0\delta(\mathbf{p}-\mathbf{p}_2) \\
& +a(\mathbf{p})a^\dagger(\mathbf{p}_1)(2\pi)^3 2p_2^0\delta(\mathbf{p}'-\mathbf{p}_2)\Big\}|0> \\
=& \frac{1}{(2\pi)^6}\int\frac{d^3\mathbf{p}}{2p^0}\int\frac{d^3\mathbf{p}'}{2p'^0}e^{-i(p.x_1+p'.x_2)} \\
& \Big\{(2\pi)^3 2p_1^0\delta(\mathbf{p}'-\mathbf{p}_1)(2\pi)^3 2p_2^0\delta(\mathbf{p}-\mathbf{p}_2) \\
& +\Big([a(\mathbf{p}),a^\dagger(\mathbf{p}_1)]+a^\dagger(\mathbf{p}_1)a(\mathbf{p})\Big)(2\pi)^3 2p_2^0\delta(\mathbf{p}'-\mathbf{p}_2)\Big\}|0> \\
=& \frac{1}{(2\pi)^6}\int\frac{d^3\mathbf{p}}{2p^0}\int\frac{d^3\mathbf{p}'}{2p'^0}e^{-i(p.x_1+p'.x_2)} \\
& \Big\{(2\pi)^3 2p_1^0\delta(\mathbf{p}'-\mathbf{p}_1)(2\pi)^3 2p_2^0\delta(\mathbf{p}-\mathbf{p}_2) \\
& +(2\pi)^3 2p_1^0\delta(\mathbf{p}-\mathbf{p}_1)(2\pi)^3 2p_2^0\delta(\mathbf{p}'-\mathbf{p}_2)\Big\}|0> \\
=& e^{-i(p_1.x_1+p_2.x_2)}|0> + e^{-i(p_2.x_1+p_1.x_2)}|0> .
\end{aligned}
$$

4.

$$
\begin{aligned}
<\mathbf{p}_3\mathbf{p}_4|\mathbf{p}_1\mathbf{p}_2> =& <0|a(\mathbf{p}_3)a(\mathbf{p}_4)a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)|0> \\
=& <0|a(\mathbf{p}_3)\left\{[a(\mathbf{p}_4),a^\dagger(\mathbf{p}_1)] + a^\dagger(\mathbf{p}_1)a(\mathbf{p}_4)\right\}a^\dagger(\mathbf{p}_2)|0> \\
=&(2\pi)^3 2p_1^0\delta(\mathbf{p}_4-\mathbf{p}_1) <0|a(\mathbf{p}_3)a^\dagger(\mathbf{p}_2)|0> \\
&+ <0|a(\mathbf{p}_3)a^\dagger(\mathbf{p}_1)a(\mathbf{p}_4)a^\dagger(\mathbf{p}_2)|0>
\end{aligned}
$$

Now

$$
\begin{aligned}
<\mathbf{p}_4|\mathbf{p}_1> =& <0|a(\mathbf{p}_4)a^\dagger(\mathbf{p}_1)|0> \\
=& <0|\left\{[a(\mathbf{p}_4),a^\dagger(\mathbf{p}_1)] + a^\dagger(\mathbf{p}_1)a(\mathbf{p}_4)\right\}|0> \\
=&(2\pi)^3 2p_1^0\delta(\mathbf{p}_4-\mathbf{p}_1) <0|0> \\
=&(2\pi)^3 2p_1^0\delta(\mathbf{p}_4-\mathbf{p}_1)
\end{aligned}
$$

so

$$
\begin{aligned}
<\mathbf{p}_3\mathbf{p}_4|\mathbf{p}_1\mathbf{p}_2> =& <\mathbf{p}_4|\mathbf{p}_1><\mathbf{p}_3|\mathbf{p}_2> \\
&+ <0|a(\mathbf{p}_3)a^\dagger(\mathbf{p}_1)\left\{[a(\mathbf{p}_4),a^\dagger(\mathbf{p}_2)] + a^\dagger(\mathbf{p}_2)a(\mathbf{p}_4)\right\}|0> \\
=& <\mathbf{p}_4|\mathbf{p}_1><\mathbf{p}_3|\mathbf{p}_2> + (2\pi)^3 2p_2^0\delta(\mathbf{p}_4-\mathbf{p}_2) <0|a(\mathbf{p}_3)a^\dagger(\mathbf{p}_1)|0> \\
=& <\mathbf{p}_4|\mathbf{p}_1><\mathbf{p}_3|\mathbf{p}_1> + <\mathbf{p}_4|\mathbf{p}_2><\mathbf{p}_3|\mathbf{p}_1> .
\end{aligned}
$$

5(a).

(b)

$$
\begin{aligned}
\Delta_F(x) =& \frac{1}{(2\pi)^4}\int_{\mathcal{C}_F}\frac{d^4k}{k^2-m^2}e^{-ik.x} \\
=& \frac{1}{(2\pi)^4}\int_{\mathcal{C}_F}\frac{d^4k}{(k^0)^2-\mathbf{k}^2-m^2}e^{-ik.x} \\
=& \frac{1}{(2\pi)^4}\int_{\mathcal{C}_F}\frac{d^4k}{(k^0-\sqrt{\mathbf{k}^2+m^2})(k^0+\sqrt{\mathbf{k}^2+m^2})}e^{-ik.x}.
\end{aligned}
$$

For $x^0>0$ we may close the contour towards $k^0\to -i\infty$.

$$
\Delta_F(x) = -\frac{1}{(2\pi)^4}\int_{\mathcal{C}^+}\frac{d^4k}{(k^0-\sqrt{\mathbf{k}^2+m^2})(k^0+\sqrt{\mathbf{k}^2+m^2})}e^{-ik.x}.
$$

The $-$ve sign is because on closing the contour $\mathcal{C}_F$ the sense of rotation is clockwise, and $\mathcal{C}^+$ is defined to be traversed anticlockwise. By Cauchy's Theorem, the value of the k^0 integral is then $(2\pi i)$ times the residue of the pole at $k^0 = \sqrt{\mathbf{k}^2 + m^2}$, and we have

$$\begin{aligned}\Delta_F(x) =& -\frac{i}{(2\pi)^3}\int \frac{d^3\mathbf{k}}{2\sqrt{\mathbf{k}^2+m^2}} e^{-i(\sqrt{\mathbf{k}^2+m^2}x^0 - \mathbf{k}.\mathbf{x})} \\ =& -\frac{i}{(2\pi)^3}\int \frac{d^3\mathbf{k}}{2k^0} e^{-ik.x},\end{aligned}$$

where in the last equation $k^0 = \sqrt{\mathbf{k}^2 + m^2}$.

For $x^0 < 0$ we may close the contour towards $k^0 \to +i\infty$.

$$\Delta_F(x) = \frac{1}{(2\pi)^4}\int_{\mathcal{C}^-} \frac{d^4k}{(k^0 - \sqrt{\mathbf{k}^2+m^2})(k^0 + \sqrt{\mathbf{k}^2+m^2})} e^{-ik.x}.$$

The value of the k^0 integral is then $(2\pi i)$ times the residue of the pole at $k^0 = -\sqrt{\mathbf{k}^2 + m^2}$, and we have

$$\begin{aligned}\Delta_F(x) =& -\frac{i}{(2\pi)^3}\int \frac{d^3\mathbf{k}}{2\sqrt{\mathbf{k}^2+m^2}} e^{-i(-\sqrt{\mathbf{k}^2+m^2}x^0 - \mathbf{k}.\mathbf{x})} \\ =& -\frac{i}{(2\pi)^3}\int \frac{d^3\mathbf{k}}{2\sqrt{\mathbf{k}^2+m^2}} e^{i(\sqrt{\mathbf{k}^2+m^2}x^0 - \mathbf{k}.\mathbf{x})} \\ =& -\frac{i}{(2\pi)^3}\int \frac{d^3\mathbf{k}}{2k^0} e^{ik.x},\end{aligned}$$

where we have used the fact that the integral over $\mathbf{k}$ is unchanged by the change $\mathbf{k} \to -\mathbf{k}$, and in the last equation $k^0 = \sqrt{\mathbf{k}^2 + m^2}$. Writing

$$\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x)$$

where

$$\begin{aligned}\phi^{(+)}(x) =& \frac{1}{(2\pi)^3}\int \frac{d^3\mathbf{p}}{2p^0} a(\mathbf{p}) e^{-ip.x} \\ \phi^{(-)}(x) =& \frac{1}{(2\pi)^3}\int \frac{d^3\mathbf{p}}{2p^0} a^\dagger(\mathbf{p}) e^{ip.x},\end{aligned}$$

we have

$$\begin{aligned}[\phi^{(+)}(x), \phi^{(-)}(0)] =& \frac{1}{(2\pi)^6}\int \frac{d^3\mathbf{p}}{2p^0}\frac{d^3\mathbf{p}'}{2p'^0}[a(\mathbf{p}), a^\dagger(\mathbf{p}')] e^{-ip.x} \\ =& \frac{1}{(2\pi)^6}\int \frac{d^3\mathbf{p}}{2p^0}\frac{d^3\mathbf{p}'}{2p'^0}(2\pi)^3 2p^0 \delta(\mathbf{p} - \mathbf{p}') e^{-ip.x} \\ =& \frac{1}{(2\pi)^3}\int \frac{d^3\mathbf{p}}{2p^0} e^{-ip.x} \\ =& \frac{1}{(2\pi)^3}\int \frac{d^3\mathbf{k}}{2k^0} e^{-ik.x},\end{aligned}$$

and similarly

$$[\phi^{(+)}(0), \phi^{(-)}(x)] = \frac{1}{(2\pi)^3} \int \frac{d^3\mathbf{k}}{2k^0} e^{ik.x}.$$

So for $x^0 > 0$

$$\Delta_F(x) = -i[\phi^{(+)}(x), \phi^{(-)}(0)],$$

while for $x^0 < 0$

$$\Delta_F(x) = -i[\phi^{(+)}(0), \phi^{(-)}(x)].$$

(c)

$$\begin{aligned} < 0|\phi(x)\phi(0)|0 > &= < 0|\phi^{(+)}(x)\phi^{(-)}(0)|0 > \\ &= < 0|[\phi^{(+)}(x), \phi^{(-)}(0)]|0 >, \end{aligned}$$

since $\phi^{(+)}|0 >= 0$, and similarly

$$< 0|\phi(0)\phi(x)|0 >=< 0|[\phi^{(+)}(0), \phi^{(-)}(x)]|0 > .$$

So

$$\begin{aligned} i\Delta_F(x) &= < 0|\phi(x)\phi(0)|0 > \quad \text{if} \quad x^0 > 0. \\ i\Delta_F(x) &= < 0|\phi(0)\phi(x)|0 > \quad \text{if} \quad x^0 < 0, \end{aligned}$$

i.e.

$$< 0|T\{\phi(x), \phi(0)\}|0 >= i\Delta_F(x).$$