

MATH425 Quantum Field Theory Solutions 10

1.

$$\begin{aligned} Q &= : \int d^3\mathbf{x} \bar{\psi} \gamma^0 \psi :=: \int d^3\mathbf{x} \psi^\dagger \gamma^0 \gamma^0 \psi : \\ &= : \int d^3\mathbf{x} \psi^\dagger \psi :=: \langle \psi, \psi \rangle : . \end{aligned}$$

Now

$$\begin{aligned} \psi(x) &= \frac{1}{(2\pi)^3} \sum_{s=1}^2 \int \frac{d^3\mathbf{p}'}{2p'^0} [\psi_{\mathbf{p}'}^{(s)}(x) a_s(\mathbf{p}') + \tilde{\psi}_{\mathbf{p}'}^{(s)}(x) b_s^\dagger(\mathbf{p}')], \\ \psi^\dagger(x) &= \frac{1}{(2\pi)^3} \sum_{r=1}^2 \int \frac{d^3\mathbf{p}}{2p^0} [\psi_{\mathbf{p}}^{(r)\dagger}(x) a_r^\dagger(\mathbf{p}) + \tilde{\psi}_{\mathbf{p}}^{(r)\dagger}(x) b_r(\mathbf{p})], \end{aligned}$$

where

$$\begin{aligned} \psi_{\mathbf{p}}^{(s)}(x) &= e^{-ip \cdot x} u_s(p), \\ \tilde{\psi}_{\mathbf{p}}^{(s)}(x) &= e^{ip \cdot x} v_s(p). \end{aligned}$$

So

$$\begin{aligned} Q &= \frac{1}{(2\pi)^3} \sum_{r=1}^2 \int \frac{d^3\mathbf{p}}{2p^0} \frac{1}{(2\pi)^3} \sum_{s=1}^2 \int \frac{d^3\mathbf{p}'}{2p'^0} \int d^3\mathbf{x} : [\psi_{\mathbf{p}}^{(r)\dagger}(x) a_r^\dagger(\mathbf{p}) + \tilde{\psi}_{\mathbf{p}}^{(r)\dagger}(x) b_r(\mathbf{p})] \\ &\quad [\psi_{\mathbf{p}'}^{(s)}(x) a_s(\mathbf{p}') + \tilde{\psi}_{\mathbf{p}'}^{(s)}(x) b_s^\dagger(\mathbf{p}')] : \\ &= \frac{1}{(2\pi)^6} \sum_{r=1}^2 \sum_{s=1}^2 \int \frac{d^3\mathbf{p}}{2p^0} \int \frac{d^3\mathbf{p}'}{2p'^0} \int d^3\mathbf{x} \\ &\quad : \left[\psi_{\mathbf{p}}^{(r)\dagger}(\mathbf{x}) \psi_{\mathbf{p}'}^{(s)}(\mathbf{x}) a_r^\dagger(\mathbf{p}) a_s(\mathbf{p}') + \psi_{\mathbf{p}}^{(r)\dagger}(\mathbf{x}) \tilde{\psi}_{\mathbf{p}'}^{(s)}(\mathbf{x}) a_r^\dagger(\mathbf{p}) b_s^\dagger(\mathbf{p}') \right. \\ &\quad \left. + \tilde{\psi}_{\mathbf{p}}^{(r)\dagger}(\mathbf{x}) \psi_{\mathbf{p}'}^{(s)}(\mathbf{x}) b_r(\mathbf{p}) a_s(\mathbf{p}') + \tilde{\psi}_{\mathbf{p}}^{(r)\dagger}(\mathbf{x}) \tilde{\psi}_{\mathbf{p}'}^{(s)}(\mathbf{x}) b_r(\mathbf{p}) b_s^\dagger(\mathbf{p}') \right] : \\ &= \frac{1}{(2\pi)^6} \sum_{r=1}^2 \sum_{s=1}^2 \int \frac{d^3\mathbf{p}}{2p^0} \int \frac{d^3\mathbf{p}'}{2p'^0} \\ &\quad : \left[\langle \psi_{\mathbf{p}}^{(r)}, \psi_{\mathbf{p}'}^{(s)} \rangle a_r^\dagger(\mathbf{p}) a_s(\mathbf{p}') + \langle \psi_{\mathbf{p}}^{(r)}, \tilde{\psi}_{\mathbf{p}'}^{(s)} \rangle a_r^\dagger(\mathbf{p}) b_s^\dagger(\mathbf{p}') \right. \\ &\quad \left. + \langle \tilde{\psi}_{\mathbf{p}}^{(r)}, \psi_{\mathbf{p}'}^{(s)} \rangle b_r(\mathbf{p}) a_s(\mathbf{p}') + \langle \tilde{\psi}_{\mathbf{p}}^{(r)}, \tilde{\psi}_{\mathbf{p}'}^{(s)} \rangle b_r(\mathbf{p}) b_s^\dagger(\mathbf{p}') \right] : . \end{aligned}$$

We have

$$\begin{aligned} \langle \psi_{\mathbf{p}}^{(r)}, \psi_{\mathbf{p}'}^{(s)} \rangle &= \langle \tilde{\psi}_{\mathbf{p}}^{(r)}, \tilde{\psi}_{\mathbf{p}'}^{(s)} \rangle = \delta_{rs} 2p^0 (2\pi)^3 \delta(\mathbf{p} - \mathbf{p}'), \\ \langle \psi_{\mathbf{p}}^{(r)}, \tilde{\psi}_{\mathbf{p}'}^{(s)} \rangle &= \langle \tilde{\psi}_{\mathbf{p}}^{(r)}, \psi_{\mathbf{p}'}^{(s)} \rangle = 0. \end{aligned}$$

So

$$\begin{aligned}
Q &= \frac{1}{(2\pi)^6} \sum_{r=1}^2 \sum_{s=1}^2 \int \frac{d^3 \mathbf{p}}{2p^0} \int \frac{d^3 \mathbf{p}'}{2p'^0} \delta_{rs} 2p^0 (2\pi)^3 \delta(\mathbf{p} - \mathbf{p}') \\
&\quad : [a_r^\dagger a_s(\mathbf{p}') + b_r(\mathbf{p}) b_s^\dagger(\mathbf{p}')] : \\
&= \frac{1}{(2\pi)^3} \sum_{r=1}^2 \int \frac{d^3 \mathbf{p}}{2p^0} : [a_r^\dagger(\mathbf{p}) a_r(\mathbf{p}) + b_r(\mathbf{p}) b_r^\dagger(\mathbf{p})] : \\
&= \frac{1}{(2\pi)^3} \sum_{r=1}^2 \int \frac{d^3 \mathbf{p}}{2p^0} [a_r^\dagger(\mathbf{p}) a_r(\mathbf{p}) - b_r^\dagger(\mathbf{p}) b_r(\mathbf{p})]
\end{aligned}$$

since : $b_r(\mathbf{p}) b_r^\dagger(\mathbf{p}) := -b_r^\dagger(\mathbf{p}) b_r(\mathbf{p})$. So

$$Q = \int d^3 \mathbf{p} [N_a(\mathbf{p}) - N_b(\mathbf{p})],$$

where

$$N_a(\mathbf{p}) = \frac{1}{(2\pi)^3} \frac{1}{2p^0} a_r^\dagger(\mathbf{p}) a_r(\mathbf{p}).$$

2(i). We write

$$(\partial\phi)^2 = \eta_{\rho\sigma} \partial^\rho \phi \partial^\sigma \phi,$$

then using

$$\frac{\partial(\partial^\rho \phi)}{\partial(\partial^\mu \phi)} = \delta^\rho_\mu,$$

we have

$$\frac{\partial}{\partial(\partial^\mu \phi)} (\partial\phi)^2 = \eta_{\rho\sigma} (\delta^\rho_\mu \partial^\sigma \phi + \partial^\rho \phi \delta^\sigma_\mu) = 2\partial_\mu \phi.$$

So equation of motion becomes

$$\partial^\mu (\partial_\mu \phi) = -m^2 \phi - \frac{1}{2} \lambda_3 \phi^2 - \frac{1}{3!} \lambda_4 \phi^3,$$

or

$$\partial^2 \phi + m^2 \phi + \frac{1}{2} \lambda_3 \phi^2 + \frac{1}{3!} \lambda_4 \phi^3 = 0.$$

Generalised momentum $\pi = \dot{\phi}$ as in free case. So we have

$$H = \int [\frac{1}{2} \dot{\phi}(x')^2 + \frac{1}{2} [\nabla \phi(x')]^2 + \frac{1}{2} m^2 \phi(x')^2 + \frac{1}{3!} \lambda_3 \phi(x')^3 + \frac{1}{4!} \lambda_4 \phi(x')^4] d^3 \mathbf{x}'.$$

We have (suppressing t)

$$\begin{aligned}
[\pi(\mathbf{x}), \phi(\mathbf{x}')^3] &= \phi(\mathbf{x}') [\pi(\mathbf{x}), \phi(\mathbf{x}')^2] + [\pi(\mathbf{x}), \phi(\mathbf{x}')] \phi(\mathbf{x}')^2 \\
&= \phi(\mathbf{x}') (\phi(\mathbf{x}') [\pi(\mathbf{x}), \phi(\mathbf{x}')] + [\pi(\mathbf{x}), \phi(\mathbf{x}')] \phi(\mathbf{x}')) + [\pi(\mathbf{x}), \phi(\mathbf{x}')] \phi(\mathbf{x}')^2 \\
&= \phi(\mathbf{x}')^2 [\pi(\mathbf{x}), \phi(\mathbf{x}')] + \phi(\mathbf{x}') [\pi(\mathbf{x}), \phi(\mathbf{x}')] \phi(\mathbf{x}') + [\pi(\mathbf{x}), \phi(\mathbf{x}')] \phi(\mathbf{x}') \\
&= -3\phi(\mathbf{x}')^2 i\hbar \delta(\mathbf{x} - \mathbf{x}').
\end{aligned}$$

Similarly,

$$[\pi(\mathbf{x}), \phi(\mathbf{x}')^4] = -4\phi(\mathbf{x}')^3 i\hbar \delta(\mathbf{x} - \mathbf{x}').$$

Writing $H = H_0 + H_I$ where

$$H_0 = \int \left[\frac{1}{2} \dot{\phi}(x')^2 + \frac{1}{2} [\nabla \phi(x')]^2 + \frac{1}{2} m^2 \phi(x')^2 \right] d^3 \mathbf{x}'$$

$$H_I = \int \left[\frac{1}{3!} \lambda_3 \phi(x')^3 + \frac{1}{4!} \lambda_4 \phi(x')^4 \right] d^3 \mathbf{x}',$$

we showed already (in the notes) that

$$[\pi(\mathbf{x}), H_0] = i\hbar [\nabla^2 \phi(\mathbf{x}) - m^2 \phi(\mathbf{x})].$$

We now have

$$[\pi(\mathbf{x}), H_I] = -i\hbar \int \left[\lambda_3 \frac{1}{2} \delta(\mathbf{x} - \mathbf{x}') \phi(\mathbf{x}')^2 + \lambda_4 \frac{1}{3!} \delta(\mathbf{x} - \mathbf{x}') \phi(\mathbf{x}')^3 \right] d^3 \mathbf{x}'$$

$$= -i\hbar \left[\frac{1}{2} \lambda_3 \phi(\mathbf{x})^2 + \frac{1}{3!} \lambda_4 \phi(\mathbf{x})^3 \right]$$

So

$$[\pi(\mathbf{x}), H] = i\hbar \left[\nabla^2 \phi(\mathbf{x}) - m^2 \phi(\mathbf{x}) - \frac{1}{2} \lambda_3 \phi(\mathbf{x})^2 - \frac{1}{3!} \lambda_4 \phi(\mathbf{x})^3 \right]$$

and

$$i\hbar \dot{\pi} = [\pi, H]$$

gives

$$\dot{\pi} = \ddot{\phi} = \nabla^2 \phi(\mathbf{x}) - m^2 \phi(\mathbf{x}) - \frac{1}{2} \lambda_3 \phi(\mathbf{x})^2 - \frac{1}{3!} \lambda_4 \phi(\mathbf{x})^3,$$

which (since $\partial^2 \phi = \ddot{\phi} - \nabla^2 \phi$) implies

$$\partial^2 \phi + m^2 \phi + \frac{1}{2} \lambda_3 \phi^2 + \frac{1}{3!} \lambda_4 \phi^3 = 0.$$

as required.