MATH425 Quantum Field Theory Solutions 6

1(a).

$$\begin{aligned}
u_r(p) =& \sqrt{p^0+m}\begin{pmatrix}\chi_r\\ \frac{\sigma.\mathbf{p}}{p^0+m}\chi_r\end{pmatrix}\\
\Rightarrow \overline{u_r(p)}u_s(p) =& u_r^\dagger\gamma^0 u_s = (p^0+m)\begin{pmatrix}\chi_r^\dagger & \chi_r^\dagger\frac{\sigma.\mathbf{p}}{p^0+m}\end{pmatrix}\begin{pmatrix}1_2 & 0_2\\ 0_2 & -1_2\end{pmatrix}\begin{pmatrix}\chi_s\\ \frac{\sigma.\mathbf{p}}{p^0+m}\chi_s\end{pmatrix}\\
=&(p^0+m)\left(\chi_r^\dagger\chi_s - \chi_r^\dagger\frac{(\sigma.\mathbf{p})^2}{(p^0+m)^2}\chi_s\right)\\
=&(p^0+m)\left(\chi_r^\dagger\chi_s - \chi_r^\dagger\frac{\mathbf{p}^2}{(p^0+m)^2}\chi_s\right)\\
=&(p^0+m)\left(1-\frac{\mathbf{p}^2}{(p^0+m)^2}\right)\chi_r^\dagger\chi_s\\
=&(p^0+m)\left(\frac{(p^0+m)^2-((p^0)^2-m^2)}{(p^0+m)^2}\right)\delta_{rs}\\
=&(p^0+m)\frac{2p^0m+2m^2}{(p^0+m)^2}\delta_{rs} = 2m\delta_{rs}.
\end{aligned}$$

(b).

$$\begin{aligned}
u_1(p)\overline{u_1(p)} =& u_1(p)u_1^\dagger\gamma^0 = (p^0+m)\begin{pmatrix}\chi_1\\ \frac{\sigma.\mathbf{p}}{p^0+m}\chi_1\end{pmatrix}\begin{pmatrix}\chi_1^\dagger & \chi_1^\dagger\frac{\sigma.\mathbf{p}}{p^0+m}\end{pmatrix}\begin{pmatrix}1_2 & 0_2\\ 0_2 & -1_2\end{pmatrix}\\
=&(p^0+m)\begin{pmatrix}\chi_1\\ \frac{\sigma.\mathbf{p}}{p^0+m}\chi_1\end{pmatrix}\begin{pmatrix}\chi_1^\dagger & -\chi_1^\dagger\frac{\sigma.\mathbf{p}}{p^0+m}\end{pmatrix}\\
=&(p^0+m)\begin{pmatrix}\chi_1\chi_1^\dagger & -\chi_1\chi_1^\dagger\frac{\sigma.\mathbf{p}}{p^0+m}\\ \frac{\sigma.\mathbf{p}}{p^0+m}\chi_1\chi_1^\dagger & -\frac{\sigma.\mathbf{p}}{p^0+m}\chi_1\chi_1^\dagger\frac{\sigma.\mathbf{p}}{p^0+m}\end{pmatrix}
\end{aligned}$$

So

$$\begin{aligned}
&u_1(p)\overline{u_1(p)} + u_2(p)\overline{u_2(p)} \\
=&(p^0+m)\begin{pmatrix} (\chi_1\chi_1^\dagger + \chi_2\chi_2^\dagger) & -(\chi_1\chi_1^\dagger + \chi_2\chi_2^\dagger)\frac{\sigma.\mathbf{p}}{p^0+m} \\ \frac{\sigma.\mathbf{p}}{p^0+m}(\chi_1\chi_1^\dagger + \chi_2\chi_2^\dagger) & -\frac{\sigma.\mathbf{p}}{p^0+m}(\chi_1\chi_1^\dagger + \chi_2\chi_2^\dagger)\frac{\sigma.\mathbf{p}}{p^0+m} \end{pmatrix} \\
=&(p^0+m)\begin{pmatrix} 1 & -\frac{\sigma.\mathbf{p}}{p^0+m} \\ \frac{\sigma.\mathbf{p}}{p^0+m} & -\frac{(\sigma.\mathbf{p})^2}{(p^0+m)^2} \end{pmatrix} \\
=&(p^0+m)\begin{pmatrix} 1 & -\frac{\sigma.\mathbf{p}}{p^0+m} \\ \frac{\sigma.\mathbf{p}}{p^0+m} & -\frac{(\mathbf{p})^2}{(p^0+m)^2} \end{pmatrix} \\
=&(p^0+m)\begin{pmatrix} 1 & -\frac{\sigma.\mathbf{p}}{p^0+m} \\ \frac{\sigma.\mathbf{p}}{p^0+m} & -\frac{(p^0)^2-m^2}{(p^0+m)^2} \end{pmatrix} \\
=&(p^0+m)\begin{pmatrix} 1 & -\frac{\sigma.\mathbf{p}}{p^0+m} \\ \frac{\sigma.\mathbf{p}}{p^0+m} & -\frac{p^0-m}{p^0+m} \end{pmatrix} \\
=&\begin{pmatrix} p^0+m & -\sigma.\mathbf{p} \\ \sigma.\mathbf{p} & -p^0+m \end{pmatrix} \\
=&\eta_{\mu\nu}\gamma^\mu p^\nu + m = \gamma.p + m.
\end{aligned}$$

$$\begin{aligned}
\text{So}\quad P_+ =&\frac{1}{2m}(u_1(p)\overline{u_1(p)} + u_2(p)\overline{u_2(p)}) \\
=&\frac{1}{2m}(\gamma.p+m).
\end{aligned}$$

(We have used $\chi_1\chi_1^\dagger + \chi_2\chi_2^\dagger = 1_2$.) So

$$\begin{aligned}
P_+^2 =&\frac{1}{4m^2}(u_1(p)\overline{u_1(p)} + u_2(p)\overline{u_2(p)})(u_1(p)\overline{u_1(p)} + u_2(p)\overline{u_2(p)}) \\
=&\frac{1}{4m^2}[(u_1(p)\overline{u_1(p)}u_1(p)\overline{u_1(p)} + u_1(p)\overline{u_1(p)}u_2(p)\overline{u_2(p)} \\
&+ u_2(p)\overline{u_2(p)}u_1(p)\overline{u_1(p)} + u_2(p)\overline{u_2(p)}u_2(p)\overline{u_2(p)}] \\
=&\frac{1}{4m^2}(u_1(p)2m\overline{u_1(p)} + 0 + 0 + u_2(p)2m\overline{u_2(p)}) \\
=&\frac{1}{2m}(u_1(p)\overline{u_1(p)} + u_2(p)\overline{u_2(p)}) = P_+.
\end{aligned}$$

Or

$$\begin{aligned}
P_+^2 =&\frac{1}{4m^2}(\gamma.p+m)^2 = \frac{1}{4m^2}[(\gamma.p)^2 + 2m\gamma.p + m^2] \\
=&\frac{1}{4m^2}[p^2 + 2m\gamma.p + m^2] = \frac{1}{4m^2}[m^2 + 2m\gamma.p + m^2] \\
=&\frac{1}{4m^2}[2m^2 + 2m\gamma.p] = \frac{1}{2m}[\gamma.p + m] = P_+.
\end{aligned}$$

Similarly,

$$
\begin{aligned}
v_1(p)\overline{v_1(p)} =& v_1(p)v_1^\dagger\gamma^0 = (p^0+m)\begin{pmatrix} \frac{\sigma.\mathbf{p}}{p^0+m}\chi_1 \\ \chi_1 \end{pmatrix}\begin{pmatrix} \chi_1^\dagger\frac{\sigma.\mathbf{p}}{p^0+m} & \chi_1^\dagger \end{pmatrix}\begin{pmatrix} 1_2 & 0_2 \\ 0_2 & -1_2 \end{pmatrix} \\
=&(p^0+m)\begin{pmatrix} \frac{\sigma.\mathbf{p}}{p^0+m}\chi_1 \\ \chi_1 \end{pmatrix}\begin{pmatrix} \chi_1^\dagger\frac{\sigma.\mathbf{p}}{p^0+m} & -\chi_1^\dagger \end{pmatrix} \\
=&(p^0+m)\begin{pmatrix} \frac{\sigma.\mathbf{p}}{p^0+m}\chi_1\chi_1^\dagger\frac{\sigma.\mathbf{p}}{p^0+m} & -\frac{\sigma.\mathbf{p}}{p^0+m}\chi_1\chi_1^\dagger \\ \chi_1\chi_1^\dagger\frac{\sigma.\mathbf{p}}{p^0+m} & -\chi_1\chi_1^\dagger \end{pmatrix}
\end{aligned}
$$

So

$$
\begin{aligned}
& v_1(p)\overline{v_1(p)} + v_2(p)\overline{v_2(p)} \\
=&(p^0+m)\begin{pmatrix} \frac{\sigma.\mathbf{p}}{p^0+m}(\chi_1\chi_1^\dagger+\chi_2\chi_2^\dagger)\frac{\sigma.\mathbf{p}}{p^0+m} & -\frac{\sigma.\mathbf{p}}{p^0+m}(\chi_1\chi_1^\dagger+\chi_2\chi_2^\dagger) \\ (\chi_1\chi_1^\dagger+\chi_2\chi_2^\dagger)\frac{\sigma.\mathbf{p}}{p^0+m} & -(\chi_1\chi_1^\dagger+\chi_2\chi_2^\dagger) \end{pmatrix} \\
=&(p^0+m)\begin{pmatrix} \frac{(\sigma.\mathbf{p})^2}{(p^0+m)^2} & -\frac{\sigma.\mathbf{p}}{p^0+m} \\ \frac{\sigma.\mathbf{p}}{p^0+m} & -1 \end{pmatrix} \\
=&(p^0+m)\begin{pmatrix} \frac{(\mathbf{p})^2}{(p^0+m)^2} & -\frac{\sigma.\mathbf{p}}{p^0+m} \\ \frac{\sigma.\mathbf{p}}{p^0+m} & -1 \end{pmatrix} \\
=&(p^0+m)\begin{pmatrix} \frac{(p^0)^2-m^2}{(p^0+m)^2} & -\frac{\sigma.\mathbf{p}}{p^0+m} \\ \frac{\sigma.\mathbf{p}}{p^0+m} & -1 \end{pmatrix} \\
=&(p^0+m)\begin{pmatrix} \frac{p^0-m}{p^0+m} & -\frac{\sigma.\mathbf{p}}{p^0+m} \\ \frac{\sigma.\mathbf{p}}{p^0+m} & -1 \end{pmatrix} \\
=&\begin{pmatrix} p^0-m & -\sigma.\mathbf{p} \\ \sigma.\mathbf{p} & -p^0-m \end{pmatrix} \\
=&\gamma.p - m.
\end{aligned}
$$

$$
\begin{aligned}
\text{So}\quad P_- =& -\frac{1}{2m}(v_1(p)\overline{v_1(p)} + v_2(p)\overline{v_2(p)}) \\
=&\frac{1}{2m}(-\gamma.p+m).
\end{aligned}
$$

So (this wasn't asked for in the question but we'll do it anyway)

$$
\begin{aligned}
P_-^2 =& \frac{1}{4m^2}(v_1(p)\overline{v_1(p)} + v_2(p)\overline{v_2(p)})(v_1(p)\overline{v_1(p)} + v_2(p)\overline{v_2(p)}) \\
=&\frac{1}{4m^2}[v_1(p)\overline{v_1(p)}v_1(p)\overline{v_1(p)} + v_1(p)\overline{v_1(p)}v_2(p)\overline{v_2(p)} \\
&+ v_2(p)\overline{v_2(p)}v_1(p)\overline{v_1(p)} + v_2(p)\overline{v_2(p)}v_2(p)\overline{v_2(p)}] \\
=&\frac{1}{4m^2}(v_1(p)(-2m)\overline{v_1(p)} + 0 + 0 + v_2(p)(-2m)\overline{v_2(p)}) \\
=&-\frac{1}{2m}(v_1(p)\overline{v_1(p)} + v_2(p)\overline{v_2(p)}) = P_-.
\end{aligned}
$$

Or

$$\begin{aligned}
P_-^2 =& \frac{1}{4m^2}(-\gamma.p + m)^2 = \frac{1}{4m^2}[(\gamma.p)^2 - 2m\gamma.p + m^2] \\
=& \frac{1}{4m^2}[p^2 - 2m\gamma.p + m^2] = \frac{1}{4m^2}[m^2 - 2m\gamma.p + m^2] \\
=& \frac{1}{4m^2}[2m^2 - 2m\gamma.p] = \frac{1}{2m}[-\gamma.p + m] = P_-.
\end{aligned}$$