MATH425 Quantum Field Theory Solutions 5

1.

$$\begin{aligned}\gamma_\mu \not{a}_1 \gamma^\mu =& a_1^\nu \gamma_\mu \gamma_\nu \gamma^\mu = a_1^\nu(-\gamma_\nu\gamma_\mu + 2\eta_{\mu\nu})\gamma^\mu \\ =& a_1^\nu(-\gamma_\nu\gamma_\mu\gamma^\mu + 2\gamma_\nu).\end{aligned}$$

Now

$$\begin{aligned}\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu =& 2\eta_{\mu\nu}, \\ \Rightarrow \eta^{\mu\nu}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) =& 2\eta^{\mu\nu}\eta_{\mu\nu} \\ \Rightarrow \gamma_\mu\gamma^\mu + \gamma_\nu\gamma^\nu =& 2\delta^\mu{}_\mu = 8 \Rightarrow 2\gamma_\mu\gamma^\mu = 8 \Rightarrow \gamma_\mu\gamma^\mu = 4.\end{aligned}$$

So we have

$$\gamma_\mu \not{a}_1 \gamma^\mu = (-4+2)a_1^\nu\gamma_\nu = -2(a_1.\gamma).$$

Also

$$\begin{aligned}\gamma_\mu \not{a}_1 \not{a}_2 \gamma^\mu =& a_1^\nu a_2^\rho \gamma_\mu\gamma_\nu\gamma_\rho\gamma^\mu \\ =& a_1^\nu a_2^\rho(-\gamma_\nu\gamma_\mu + 2\eta_{\mu\nu})\gamma_\rho\gamma^\mu \\ =& a_1^\nu a_2^\rho(-\gamma_\nu\gamma_\mu\gamma_\rho\gamma^\mu + 2\gamma_\rho\gamma_\nu.\end{aligned}$$

Now we know that $\gamma_\mu\gamma_\rho\gamma^\mu = -2\gamma_\rho$, so we have

$$\gamma_\mu \not{a}_1 \not{a}_2 \gamma^\mu = 2a_1^\nu a_2^\rho(\gamma_\nu\gamma_\rho + \gamma_\rho\gamma_\nu) = 4a_1^\nu a_2^\rho \eta_{\nu\rho} = 4a_1.a_2.$$

2. Under a Lorentz transformation we have

$$\begin{aligned}j'^\mu(x') =& \overline{\psi'(x')}\gamma^5\gamma^\mu\psi'(x') = \overline{\psi(x)}S(L)^{-1}\gamma^5\gamma^\mu S(L)\psi(x) \\ =& \overline{\psi(x)}S(L)^{-1}\gamma^5 S(L)S(L)^{-1}\gamma^\mu S(L)\psi(x) = L^\mu{}_\nu\overline{\psi(x)}\gamma^5\gamma^\nu\psi(x) = L^\mu{}_\nu j^\nu(x),\end{aligned}$$

using $S(L)^{-1}\gamma^5 S(L) = \gamma^5$, $S(L)^{-1}\gamma^\mu S(L) = L^\mu{}_\nu\gamma^\nu$. Under a parity transformation we have

$$\begin{aligned}j'^\mu(x') =& \overline{\psi'(x')}\gamma^5\gamma^\mu\psi'(x') = \overline{\psi(x)}S(P)^{-1}\gamma^5\gamma^\mu S(P)\psi(x) \\ =& \overline{\psi(x)}S(P)^{-1}\gamma^5 S(P)S(P)^{-1}\gamma^\mu S(P)\psi(x) \\ =& -P^\mu{}_\nu\overline{\psi(x)}\gamma^5\gamma^\nu\psi(x) = -P^\mu{}_\nu j^\nu(x),\end{aligned}$$

using $S(P)^{-1}\gamma^5 S(P) = -\gamma^5$, $S(P)^{-1}\gamma^\mu S(P) = P^\mu{}_\nu\gamma^\nu$.

3. Consider the plane wave $\psi(x) = v(p)e^{ip.x}$. It will satisfy the Dirac equation

$$\begin{aligned} (i\gamma.\partial - m)\psi =& 0 \\ \text{if} \quad (\gamma.p + m)v(p) =& 0. \end{aligned}$$

In our representation,

$$\gamma.p + m = \begin{pmatrix} p^0 + m & -\sigma.\mathbf{p} \\ \sigma.\mathbf{p} & -p^0 + m \end{pmatrix}$$

where each entry is a 2×2 block. Now write

$$v(p) = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \quad \text{where} \quad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \quad \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}.$$

Then

$$(\gamma.p + m)v(p) = 0$$

becomes

$$\begin{aligned} (p^0 + m)\xi =& \sigma.\mathbf{p}\eta, \\ \sigma.\mathbf{p}\xi =& (p^0 - m)\eta. \end{aligned}$$

Given η, define

$$\begin{aligned} \xi =& \frac{\sigma.\mathbf{p}}{p^0 + m}\eta \\ \text{then} \quad \sigma.\mathbf{p}\xi =& \frac{(\sigma.\mathbf{p})^2}{p^0 + m}\eta \\ \text{Now} \quad \sigma^i\sigma^j p_i p_j =& \mathbf{p}^2 \quad (\sigma^i\sigma^j = \delta_{ij} + i\epsilon_{ijk}\sigma^k) \\ \text{So} \quad \sigma.\mathbf{p}\xi =& \frac{\mathbf{p}^2}{p^0 + m}\eta \\ =& \frac{(p^0)^2 - m^2}{p^0 + m}\eta \\ =& (p^0 - m)\eta \end{aligned}$$

so we have a consistent solution. The general form of $v(p)$ is

$$v(p) = N(p) \begin{pmatrix} \frac{\sigma.\mathbf{p}}{p^0+m}\chi \\ \chi \end{pmatrix},$$

where χ is a 2-vector and $N(p)$ is arbitrary. Suppose we pick two orthonormal 2-vectors $\chi_{1,2}$ satisfying

$$\chi_r^\dagger \chi_s = \delta_{rs}.$$

We have

$$
\begin{aligned}
\overline{v_r(p)}v_s(p) = v_r^\dagger(p)\gamma^0 v_s(p) =& N(p)^* N(p) \begin{pmatrix} \chi_r^\dagger \frac{\sigma.\mathbf{p}}{p^0+m} & \chi_r^\dagger \end{pmatrix} \gamma^0 \begin{pmatrix} \frac{\sigma.\mathbf{p}}{p^0+m}\chi_s \\ \chi_s \end{pmatrix} \\
=& N(p)^* N(p) \begin{pmatrix} \chi_r^\dagger \frac{\sigma.\mathbf{p}}{p^0+m} & \chi_r^\dagger \end{pmatrix} \begin{pmatrix} 1_2 & 0_2 \\ 0_2 & -1_2 \end{pmatrix} \begin{pmatrix} \frac{\sigma.\mathbf{p}}{p^0+m}\chi_s \\ \chi_s \end{pmatrix} \\
=& |N(p)|^2 \left[\chi_r^\dagger \frac{(\sigma.\mathbf{p})^2}{(p^0+m)^2}\chi_s - \chi_r^\dagger \chi_s \right] \\
=& |N(p)|^2 \left[\chi_r^\dagger \frac{\mathbf{p}^2}{(p^0+m)^2}\chi_s - \chi_r^\dagger \chi_s \right] \\
=& |N(p)|^2 \chi_r^\dagger \chi_s \frac{\mathbf{p}^2 - (p^0+m)^2}{(p^0+m)^2} \\
=& |N(p)|^2 \delta_{rs} \frac{(p^0)^2 - m^2 - (p^0+m)^2}{(p^0+m)^2} \\
=& |N(p)|^2 \delta_{rs} \frac{-2p^0 m - 2m^2}{(p^0+m)^2} = -2m\delta_{rs}
\end{aligned}
$$

if we take $N(p) = \sqrt{p^0+m}$.