MATH425 Quantum Field Theory Solutions 4

1.

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \Rightarrow \{\gamma^\mu, \gamma_\nu\} = 2\delta^\mu{}_\nu.$$

So

$$\begin{aligned}\gamma^\mu\gamma_\lambda\gamma_\rho =& -\gamma_\lambda\gamma^\mu\gamma_\rho + 2\delta^\mu{}_\lambda\gamma_\rho \\ =&\gamma_\lambda\gamma_\rho\gamma^\mu - 2\delta^\mu{}_\rho\gamma_\lambda + 2\delta^\mu{}_\lambda\gamma_\rho \\ \Rightarrow [\gamma^\mu, \gamma_\lambda\gamma_\rho] =& 2(\delta^\mu{}_\lambda\gamma_\rho - \delta^\mu{}_\rho\gamma_\lambda).\end{aligned}$$

So

$$\begin{aligned}[\gamma^\mu, \gamma_\rho\gamma_\lambda] =& 2(\delta^\mu{}_\rho\gamma_\lambda - \delta^\mu{}_\lambda\gamma_\rho) \\ \text{and} \quad [\gamma^\mu, \gamma_\lambda\gamma_\rho - \gamma_\rho\gamma_\lambda] =& 4(\delta^\mu{}_\lambda\gamma_\rho - \delta^\mu{}_\rho\gamma_\lambda).\end{aligned}$$

Multiplying by $\frac{1}{8}$, we can write as

$$\begin{aligned}i[\gamma^\mu, -\tfrac{1}{8}i[\gamma_\lambda, \gamma_\rho]] =& \tfrac{1}{2}(\delta^\mu{}_\lambda\gamma_\rho - \delta^\mu{}_\rho\gamma_\lambda), \\ \text{i.e.} \quad i[\gamma^\mu, \Sigma_{\lambda\rho}] =& \tfrac{1}{2}(\delta^\mu{}_\lambda\gamma_\rho - \delta^\mu{}_\rho\gamma_\lambda).\end{aligned}$$

where

$$\Sigma_{\lambda\rho} = -\tfrac{1}{8}i[\gamma_\lambda, \gamma_\rho].$$

(Sorry about sign mistake in question.)

2.

$$\begin{aligned}\gamma^{5\dagger} =& (i\gamma^0\gamma^1\gamma^2\gamma^3)^\dagger \\ =& -i\gamma^{3\dagger}\gamma^{2\dagger}\gamma^{1\dagger}\gamma^{0\dagger} \\ =& -i(\gamma^0\gamma^3\gamma^0)(\gamma^0\gamma^2\gamma^0)(\gamma^0\gamma^1\gamma^0)\gamma^0 \\ =& -i\gamma^0\gamma^3\gamma^2\gamma^1 = i\gamma^0\gamma^2\gamma^3\gamma^1 = -i\gamma^0\gamma^2\gamma^1\gamma^3 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5.\end{aligned}$$

For the second part, it's best to do for each μ in turn:

$$\begin{aligned}\gamma^5\gamma^0 =& i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0 = -i\gamma^0\gamma^1\gamma^2\gamma^0\gamma^3 = i\gamma^0\gamma^1\gamma^0\gamma^2\gamma^3 \\ =& -i\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^0\gamma^5 \Rightarrow \{\gamma^5, \gamma^0\} = 0, \\ \gamma^5\gamma^1 =& i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1 = -i\gamma^0\gamma^1\gamma^2\gamma^1\gamma^3 = i\gamma^0\gamma^1\gamma^1\gamma^2\gamma^3 \\ =& -i\gamma^1\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^1\gamma^5 \Rightarrow \{\gamma^5, \gamma^1\} = 0.\end{aligned}$$

It is clear that the other two calculations will be similar.

3. N.B. Of course the question was wrong–should have said $(\gamma^0)^2 = 1$, $(\gamma^i)^2 = -1$, $i = 1, 2, 3$.

$$\begin{aligned}\gamma^0\gamma^1\gamma^2 =& -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^3 = i\gamma^5\gamma^3 \\ \gamma^0\gamma^1\gamma^3 =& -\gamma^0\gamma^1\gamma^2\gamma^2\gamma^3 = \gamma^0\gamma^1\gamma^2\gamma^3\gamma^2 = -i\gamma^5\gamma^2.\end{aligned}$$

4.

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}1_4 \Rightarrow \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}1_4,$$

where 1_4 is the 4-dimensional identity matrix, usually not written explicitly. Taking the trace, and using $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, $\mathrm{tr}1_4 = 4$, we get

$$\mathrm{tr}[\gamma_\mu \gamma_\nu] = 4\eta_{\mu\nu}.$$

Now

$$\begin{aligned} \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma &= -\gamma_\nu \gamma_\mu \gamma_\rho \gamma_\sigma + 2\eta_{\mu\nu} \gamma_\rho \gamma_\sigma \\ &= \gamma_\nu \gamma_\rho \gamma_\mu \gamma_\sigma - 2\eta_{\mu\rho} \gamma_\nu \gamma_\sigma + 2\eta_{\mu\nu} \gamma_\rho \gamma_\sigma \\ &= -\gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\mu + 2\eta_{\mu\sigma} \gamma_\nu \gamma_\rho - 2\eta_{\mu\rho} \gamma_\nu \gamma_\sigma + 2\eta_{\mu\nu} \gamma_\rho \gamma_\sigma \\ \Rightarrow \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma + \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\mu &= 2\eta_{\mu\sigma} \gamma_\nu \gamma_\rho - 2\eta_{\mu\rho} \gamma_\nu \gamma_\sigma + 2\eta_{\mu\nu} \gamma_\rho \gamma_\sigma. \end{aligned}$$

Taking the trace and using

$$\mathrm{tr}[\gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\mu] = \mathrm{tr}[\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma],$$

together with $\mathrm{tr}[\gamma_\mu \gamma_\nu = 4\eta_{\mu\nu}$, we find

$$\mathrm{tr}[\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma] = 4[\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}]$$