

MATH425 Quantum Field Theory Solutions 2

1.

$$L = \begin{pmatrix} \gamma & -\frac{\gamma v}{c} \\ -\frac{\gamma v}{c} & \gamma \end{pmatrix}$$

$$\Rightarrow L^T = L.$$

$$\begin{aligned} \text{So } L^T \eta L &= \begin{pmatrix} \gamma & -\frac{\gamma v}{c} \\ -\frac{\gamma v}{c} & \gamma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \gamma & -\frac{\gamma v}{c} \\ -\frac{\gamma v}{c} & \gamma \end{pmatrix} \\ &= \begin{pmatrix} \gamma & -\frac{\gamma v}{c} \\ -\frac{\gamma v}{c} & \gamma \end{pmatrix} \begin{pmatrix} \gamma & -\frac{\gamma v}{c} \\ \frac{\gamma v}{c} & -\gamma \end{pmatrix} \\ &= \begin{pmatrix} \gamma^2 - \left(\frac{\gamma v}{c}\right)^2 & -\gamma\frac{\gamma v}{c} + \gamma\frac{\gamma v}{c} \\ -\gamma\frac{\gamma v}{c} + \gamma\frac{\gamma v}{c} & \left(\frac{\gamma v}{c}\right)^2 - \gamma^2 \end{pmatrix} \\ &= \begin{pmatrix} \gamma^2 \left(1 - \frac{v^2}{c^2}\right) & 0 \\ 0 & -\gamma^2 \left(1 - \frac{v^2}{c^2}\right) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \eta, \end{aligned}$$

since

$$\gamma^2 = \frac{1}{1 - \frac{v^2}{c^2}}.$$

2(i). Taking the inverse of (1), we have

$$(L^T \eta L)^{-1} = \eta^{-1} \Rightarrow L^{-1} \eta^{-1} (L^T)^{-1} = \eta^{-1}$$

Multiplying on the left by L and on the right by L^T , we find

$$\begin{aligned} L L^{-1} \eta^{-1} (L^T)^{-1} L^T &= L \eta^{-1} L^T \\ \eta^{-1} &= L \eta^{-1} L^T, \end{aligned}$$

as required.

(ii) Putting $\mu = \nu = 0$ in (3), we have

$$\begin{aligned} \eta_{\alpha\beta} L^\alpha_0 L^\beta_0 &= 1, \quad \eta^{\alpha\beta} L^0_\alpha L^0_\beta = 1 \\ \text{i.e. } (L^0_0)^2 - (L^1_0)^2 - (L^2_0)^2 - (L^3_0)^2 &= 1, \\ (\bar{L}^0_0)^2 - (\bar{L}^0_1)^2 - (\bar{L}^0_2)^2 - (\bar{L}^0_3)^2 &= 1 \\ \Rightarrow (L^0_0)^2 - |\mathbf{l}|^2 &= 1, \quad (\bar{L}^0_0)^2 - |\bar{\mathbf{l}}|^2 = 1 \\ \Rightarrow |\mathbf{l}| &= \sqrt{(L^0_0)^2 - 1}, \quad |\bar{\mathbf{l}}| = \sqrt{(\bar{L}^0_0)^2 - 1}. \end{aligned}$$

(iii)

$$\begin{aligned} (\bar{L}L)^0_0 &= \bar{L}^0_\alpha L^\alpha_0 \\ &= \bar{L}^0_0 L^0_0 + \bar{L}^0_1 L^1_0 + \bar{L}^0_2 L^2_0 + \bar{L}^0_3 L^3_0 \\ &= \bar{L}^0_0 L^0_0 + \bar{\mathbf{l}} \cdot \mathbf{l} \end{aligned}$$

(iv)

$$\begin{aligned}
& |\bar{\mathbf{1}}\mathbf{1}| \leq |\mathbf{1}||\bar{\mathbf{1}}| \Rightarrow -|\mathbf{1}||\bar{\mathbf{1}}| \leq \bar{\mathbf{1}}\mathbf{1} \leq |\mathbf{1}||\bar{\mathbf{1}}| \\
\Rightarrow & \text{ (using (iii)) } (\bar{L}L)^0_0 - \bar{L}^0_0 L^0_0 \geq -|\mathbf{1}||\bar{\mathbf{1}}| \\
& \Rightarrow (\bar{L}L)^0_0 \geq \bar{L}^0_0 L^0_0 - |\mathbf{1}||\bar{\mathbf{1}}| \\
& \text{i.e. (using (ii)) } (\bar{L}L)^0_0 \geq \bar{L}^0_0 L^0_0 - \sqrt{(L^0_0)^2 - 1} \sqrt{(\bar{L}^0_0)^2 - 1}.
\end{aligned}$$

(v)

$$\begin{aligned}
(x-y)^2 \geq 0 & \Rightarrow x^2 - 2xy + y^2 \geq 0 \Rightarrow -2xy \geq -x^2 - y^2 \\
& \Rightarrow x^2 y^2 - 2xy + 1 \geq x^2 y^2 - x^2 - y^2 + 1 \\
& \Rightarrow x^2 y^2 - 2xy + 1 \geq (x^2 - 1)(y^2 - 1) \Rightarrow (xy - 1)^2 \geq (x^2 - 1)(y^2 - 1). \\
& \Rightarrow \text{either } xy - 1 \geq \sqrt{x^2 - 1} \sqrt{y^2 - 1} \quad \text{or} \quad xy - 1 \leq -\sqrt{x^2 - 1} \sqrt{y^2 - 1}.
\end{aligned}$$

If $x, y \geq 1$ then $xy - 1$ is positive, and we must have the first inequality, implying

$$xy - \sqrt{x^2 - 1} \sqrt{y^2 - 1} \geq 1.$$

Writing $L^0_0 = x$, $\bar{L}^0_0 = y$, we have from (iv)

$$(\bar{L}L)^0_0 \geq \bar{L}^0_0 L^0_0 - \sqrt{(L^0_0)^2 - 1} \sqrt{(\bar{L}^0_0)^2 - 1} \geq 1.$$

(vi) Obviously if $\det \bar{L} = \det L = 1$, then

$$\det(\bar{L}L) = \det \bar{L} \det L = 1.$$

(vii) We have now shown that if $L \in \mathcal{L}_+^\uparrow$, $L \in \mathcal{L}_+^\uparrow$, then $\bar{L}L \in \mathcal{L}_+^\uparrow$. It is clear that $1 \in \mathcal{L}_+^\uparrow$.

(viii) We can write (1) as

$$(\eta^{-1} L^T \eta) L = \eta^{-1} \eta = 1,$$

which shows that $L^{-1} = \eta^{-1} L^T \eta$, i.e. $(L^{-1})^\mu{}_\nu = \eta^{\mu\alpha} (L^T)_\alpha{}^\beta \eta_{\beta\nu} = \eta^{\mu\alpha} L^\beta{}_\alpha \eta_{\beta\nu}$. So $(L^{-1})^0_0 = \eta^{00} L^0_0 \eta_{00} = L^0_0$. Moreover,

$$\det L^{-1} = \det \eta^{-1} \det L^T \det \eta = (-1) \det L(-1) = \det L = 1.$$

So $L^{-1} \in \mathcal{L}_+^\uparrow$. The remaining group property is associativity, $(L_1 L_2) L_3 = L_1 (L_2 L_3)$, which is true for all matrices. So $\mathcal{L}_+^\uparrow$ is a group.