

MATH425 Quantum Field Theory Solutions 10

1.

$$\begin{aligned}
 [P^\mu, \phi(x)] &= \left[\frac{1}{(2\pi)^3} \int d^3\mathbf{p} \frac{1}{2p^0} p^\mu a^\dagger(\mathbf{p}) a(\mathbf{p}), \right. \\
 &\quad \left. \frac{1}{(2\pi)^3} \int d^3\mathbf{p}' \frac{1}{2p'^0} \left\{ a(\mathbf{p}') e^{-ip' \cdot x} + a^\dagger(\mathbf{p}') e^{ip' \cdot x} \right\} \right] \\
 &= \frac{1}{(2\pi)^6} \int d^3\mathbf{p} d^3\mathbf{p}' \frac{1}{2p^0} \frac{1}{2p'^0} p^\mu \left\{ [a^\dagger(\mathbf{p}) a(\mathbf{p}), a(\mathbf{p}')] e^{-ip' \cdot x} \right. \\
 &\quad \left. + [a^\dagger(\mathbf{p}) a(\mathbf{p}), a^\dagger(\mathbf{p}')] e^{ip' \cdot x} \right\}.
 \end{aligned}$$

Now

$$\begin{aligned}
 [a^\dagger(\mathbf{p}) a(\mathbf{p}), a(\mathbf{p}')] &= a^\dagger(\mathbf{p}) [a(\mathbf{p}), a(\mathbf{p}')] + [a^\dagger(\mathbf{p}), a(\mathbf{p}')] a(\mathbf{p}) \\
 &= - (2\pi)^3 2p'^0 \delta(\mathbf{p} - \mathbf{p}') a(\mathbf{p}), \\
 [a^\dagger(\mathbf{p}) a(\mathbf{p}), a^\dagger(\mathbf{p}')] &= a^\dagger(\mathbf{p}) [a(\mathbf{p}), a^\dagger(\mathbf{p}')] + [a^\dagger(\mathbf{p}), a^\dagger(\mathbf{p}')] a(\mathbf{p}) \\
 &= a^\dagger(\mathbf{p}) (2\pi)^3 2p'^0 \delta(\mathbf{p} - \mathbf{p}').
 \end{aligned}$$

So

$$\begin{aligned}
 [P^\mu, \phi(x)] &= \frac{1}{(2\pi)^3} \int d^3\mathbf{p} \frac{1}{2p^0} p^\mu [-a(\mathbf{p}) e^{-ip \cdot x} + a^\dagger(\mathbf{p}) e^{ip \cdot x}] \\
 &= -i\partial^\mu \phi(x).
 \end{aligned}$$

So

$$\begin{aligned}
 e^{-ia \cdot P} \phi(x) e^{ia \cdot P} &= (1 - ia \cdot P) \phi(x) (1 + ia \cdot P) + O(a^2) \\
 &= \phi(x) + i\phi(x) a \cdot P - ia \cdot P \phi(x) + O(a^2) \\
 &= \phi(x) + ia_\mu (\phi(x) P^\mu - P^\mu \phi(x)) + O(a^2) \\
 &= \phi(x) - ia_\mu [P^\mu, \phi] + O(a^2) \\
 &= \phi(x) - a_\mu \partial^\mu \phi(x) + O(a^2) = \phi(x - a)
 \end{aligned}$$

using Taylor's Theorem to 1st order.

2.

$$\begin{aligned}
\phi(x) &= \frac{1}{(2\pi)^3} \int d^3\mathbf{p} \frac{1}{2p^0} [a(\mathbf{p})e^{-ip \cdot x} + a^\dagger(\mathbf{p})e^{ip \cdot x}] \\
\Rightarrow \pi(x) = \dot{\phi}(x) &= \frac{1}{(2\pi)^3} \int d^3\mathbf{p} \frac{1}{2p^0} [-ip^0 a(\mathbf{p})e^{-ip \cdot x} + ip^0 a^\dagger(\mathbf{p})e^{ip \cdot x}] \\
\Rightarrow \int [p'^0 \phi(x) - i\pi(x)] e^{-ip' \cdot x} d^3\mathbf{x} &= \frac{1}{(2\pi)^3} \int d^3\mathbf{p} [(p'^0 - p^0) a(\mathbf{p}) e^{-i(p'^0 + p^0)x^0} e^{i(\mathbf{p} + \mathbf{p}') \cdot \mathbf{x}} \\
&\quad + (p'^0 + p^0) a^\dagger(\mathbf{p}) e^{i(p^0 - p'^0)x^0} e^{-i(\mathbf{p} - \mathbf{p}') \cdot \mathbf{x}}] d^3\mathbf{x} \\
&= \frac{1}{(2\pi)^3} \int d^3\mathbf{p} [(p'^0 - p^0) a(\mathbf{p}) e^{-i(p'^0 + p^0)x^0} (2\pi)^3 \delta(\mathbf{p} + \mathbf{p}') \\
&\quad + (p'^0 + p^0) a^\dagger(\mathbf{p}) e^{i(p^0 - p'^0)x^0} (2\pi)^3 \delta(\mathbf{p} - \mathbf{p}')] d^3\mathbf{x} \\
&= a^\dagger(\mathbf{p}') \\
\text{and similarly } \int [p^0 \phi(x) + i\pi(x)] e^{ip' \cdot x} d^3\mathbf{x} &= a(\mathbf{p}').
\end{aligned}$$

Then

$$\begin{aligned}
[a(\mathbf{p}), a^\dagger(\mathbf{p}')] &= \left[\int [p^0 \phi(x) + i\pi(x)] e^{ip \cdot x} d^3\mathbf{x}, \int [p'^0 \phi(x') - i\pi(x')] e^{-ip' \cdot x'} d^3\mathbf{x}' \right] \\
&= \int d^3\mathbf{x} d^3\mathbf{x}' \{ -ip^0 [\phi(x), \pi(x')] + ip'^0 [\pi(x), \phi(x')] \} e^{ip \cdot x} e^{-ip' \cdot x'}
\end{aligned}$$

Now x^0 and x'^0 are arbitrary, so we can choose $x^0 = x'^0 = t$ say, in order to use the equal time commutation relations. Then

$$\begin{aligned}
[a(\mathbf{p}), a^\dagger(\mathbf{p}')] &= \int d^3\mathbf{x} d^3\mathbf{x}' \{ -ip^0 [\phi(\mathbf{x}, t), \pi(\mathbf{x}', t)] + ip'^0 [\pi(\mathbf{x}, t), \phi(\mathbf{x}', t)] \} e^{ip \cdot x} e^{-ip' \cdot x'} \\
&= \int d^3\mathbf{x} d^3\mathbf{x}' [p^0 + p'^0] \delta(\mathbf{x} - \mathbf{x}') e^{ip \cdot x} e^{-ip' \cdot x'} \\
&= \int d^3\mathbf{x} [p^0 + p'^0] e^{i(p^0 - p'^0)t} e^{i(\mathbf{p}' - \mathbf{p}) \cdot \mathbf{x}} \\
&= (2\pi)^3 \delta(\mathbf{p} - \mathbf{p}') [p^0 + p'^0] e^{i(p^0 - p'^0)t} \\
&= (2\pi)^3 2p^0 \delta(\mathbf{p} - \mathbf{p}')
\end{aligned}$$

Also

$$\begin{aligned}
[a^\dagger(\mathbf{p}), a^\dagger(\mathbf{p}')] &= \left[\int [p^0 \phi(x) - i\pi(x)] e^{-ip \cdot x} d^3 \mathbf{x}, \int [p'^0 \phi(x') - i\pi(x')] e^{-ip' \cdot x'} d^3 \mathbf{x}' \right] \\
&= \int d^3 \mathbf{x} d^3 \mathbf{x}' \{ ip^0 [\phi(\mathbf{x}, t), \pi(\mathbf{x}', t)] + ip'^0 [\pi(\mathbf{x}, t), \phi(\mathbf{x}', t)] \} e^{-ip \cdot x} e^{-ip' \cdot x'} \\
&= \int d^3 \mathbf{x} d^3 \mathbf{x}' [-p^0 + p'^0] \delta(\mathbf{x} - \mathbf{x}') e^{-ip \cdot x} e^{-ip' \cdot x'} \\
&= \int d^3 \mathbf{x} [-p^0 + p'^0] e^{-i(p^0 + p'^0)t} e^{i(\mathbf{p}' + \mathbf{p}) \cdot \mathbf{x}} \\
&= (2\pi)^3 \delta(\mathbf{p} + \mathbf{p}') [-p^0 + p'^0] e^{-i(p^0 + p'^0)t} \\
&= 0.
\end{aligned}$$

3.

$$\begin{aligned}
\phi^{(+)}(x_1)\phi^{(+)}(x_2)|\mathbf{p}_1\mathbf{p}_2 > &= \frac{1}{(2\pi)^3} \int \frac{d^3\mathbf{p}}{2p^0} a(\mathbf{p}) e^{-ip \cdot x_1} \frac{1}{(2\pi)^3} \int \frac{d^3\mathbf{p}'}{2p'^0} a(\mathbf{p}') e^{-ip' \cdot x_2} \\
&\quad a^\dagger(\mathbf{p}_1) a^\dagger(\mathbf{p}_2) |0 > \\
&= \frac{1}{(2\pi)^6} \int \frac{d^3\mathbf{p}}{2p^0} \int \frac{d^3\mathbf{p}'}{2p'^0} e^{-i(p \cdot x_1 + p' \cdot x_2)} \\
&\quad a(\mathbf{p}) a(\mathbf{p}') a^\dagger(\mathbf{p}_1) a^\dagger(\mathbf{p}_2) |0 > \\
&= \frac{1}{(2\pi)^6} \int \frac{d^3\mathbf{p}}{2p^0} \int \frac{d^3\mathbf{p}'}{2p'^0} e^{-i(p \cdot x_1 + p' \cdot x_2)} \\
&\quad a(\mathbf{p}) \left\{ [a(\mathbf{p}'), a^\dagger(\mathbf{p}_1)] + a^\dagger(\mathbf{p}_1) a(\mathbf{p}') \right\} a^\dagger(\mathbf{p}_2) |0 > \\
&= \frac{1}{(2\pi)^6} \int \frac{d^3\mathbf{p}}{2p^0} \int \frac{d^3\mathbf{p}'}{2p'^0} e^{-i(p \cdot x_1 + p' \cdot x_2)} \\
&\quad \left\{ (2\pi)^3 2p_1^0 \delta(\mathbf{p}' - \mathbf{p}_1) a(\mathbf{p}') a^\dagger(\mathbf{p}_2) + a(\mathbf{p}) a^\dagger(\mathbf{p}_1) a(\mathbf{p}') a^\dagger(\mathbf{p}_2) \right\} |0 > \\
&= \frac{1}{(2\pi)^6} \int \frac{d^3\mathbf{p}}{2p^0} \int \frac{d^3\mathbf{p}'}{2p'^0} e^{-i(p \cdot x_1 + p' \cdot x_2)} \\
&\quad \left\{ (2\pi)^3 2p_1^0 \delta(\mathbf{p}' - \mathbf{p}_1) \left([a(\mathbf{p}), a^\dagger(\mathbf{p}_2)] + a^\dagger(\mathbf{p}_2) a(\mathbf{p}) \right) \right. \\
&\quad \left. + a(\mathbf{p}) a^\dagger(\mathbf{p}_1) \left([a(\mathbf{p}'), a^\dagger(\mathbf{p}_2)] + a^\dagger(\mathbf{p}_2) a(\mathbf{p}') \right) \right\} |0 > \\
&= \frac{1}{(2\pi)^6} \int \frac{d^3\mathbf{p}}{2p^0} \int \frac{d^3\mathbf{p}'}{2p'^0} e^{-i(p \cdot x_1 + p' \cdot x_2)} \\
&\quad \left\{ (2\pi)^3 2p_1^0 \delta(\mathbf{p}' - \mathbf{p}_1) (2\pi)^3 2p_2^0 \delta(\mathbf{p} - \mathbf{p}_2) \right. \\
&\quad \left. + a(\mathbf{p}) a^\dagger(\mathbf{p}_1) (2\pi)^3 2p_2^0 \delta(\mathbf{p}' - \mathbf{p}_2) \right\} |0 > \\
&= \frac{1}{(2\pi)^6} \int \frac{d^3\mathbf{p}}{2p^0} \int \frac{d^3\mathbf{p}'}{2p'^0} e^{-i(p \cdot x_1 + p' \cdot x_2)} \\
&\quad \left\{ (2\pi)^3 2p_1^0 \delta(\mathbf{p}' - \mathbf{p}_1) (2\pi)^3 2p_2^0 \delta(\mathbf{p} - \mathbf{p}_2) \right. \\
&\quad \left. + \left([a(\mathbf{p}), a^\dagger(\mathbf{p}_1)] + a^\dagger(\mathbf{p}_1) a(\mathbf{p}) \right) (2\pi)^3 2p_2^0 \delta(\mathbf{p}' - \mathbf{p}_2) \right\} |0 > \\
&= \frac{1}{(2\pi)^6} \int \frac{d^3\mathbf{p}}{2p^0} \int \frac{d^3\mathbf{p}'}{2p'^0} e^{-i(p \cdot x_1 + p' \cdot x_2)} \\
&\quad \left\{ (2\pi)^3 2p_1^0 \delta(\mathbf{p}' - \mathbf{p}_1) (2\pi)^3 2p_2^0 \delta(\mathbf{p} - \mathbf{p}_2) \right. \\
&\quad \left. + (2\pi)^3 2p_1^0 \delta(\mathbf{p} - \mathbf{p}_1) (2\pi)^3 2p_2^0 \delta(\mathbf{p}' - \mathbf{p}_2) \right\} |0 > \\
&= e^{-i(p_1 \cdot x_1 + p_2 \cdot x_2)} |0 > + e^{-i(p_2 \cdot x_1 + p_1 \cdot x_2)} |0 > .
\end{aligned}$$

4.

$$\begin{aligned}
\langle \mathbf{p}_3 \mathbf{p}_4 | \mathbf{p}_1 \mathbf{p}_2 \rangle &= \langle 0 | a(\mathbf{p}_3) a(\mathbf{p}_4) a^\dagger(\mathbf{p}_1) a^\dagger(\mathbf{p}_2) | 0 \rangle \\
&= \langle 0 | a(\mathbf{p}_3) \{ [a(\mathbf{p}_4), a^\dagger(\mathbf{p}_1)] + a^\dagger(\mathbf{p}_1) a(\mathbf{p}_4) \} a^\dagger(\mathbf{p}_2) | 0 \rangle \\
&= (2\pi)^3 2p_1^0 \delta(\mathbf{p}_4 - \mathbf{p}_1) \langle 0 | a(\mathbf{p}_3) a^\dagger(\mathbf{p}_2) | 0 \rangle \\
&\quad + \langle 0 | a(\mathbf{p}_3) a^\dagger(\mathbf{p}_1) a(\mathbf{p}_4) a^\dagger(\mathbf{p}_2) | 0 \rangle
\end{aligned}$$

Now

$$\begin{aligned}
\langle \mathbf{p}_4 | \mathbf{p}_1 \rangle &= \langle 0 | a(\mathbf{p}_4) a^\dagger(\mathbf{p}_1) | 0 \rangle \\
&= \langle 0 | \{ [a(\mathbf{p}_4), a^\dagger(\mathbf{p}_1)] + a^\dagger(\mathbf{p}_1) a(\mathbf{p}_4) \} | 0 \rangle \\
&= (2\pi)^3 2p_1^0 \delta(\mathbf{p}_4 - \mathbf{p}_1) \langle 0 | 0 \rangle \\
&= (2\pi)^3 2p_1^0 \delta(\mathbf{p}_4 - \mathbf{p}_1)
\end{aligned}$$

so

$$\begin{aligned}
\langle \mathbf{p}_3 \mathbf{p}_4 | \mathbf{p}_1 \mathbf{p}_2 \rangle &= \langle \mathbf{p}_4 | \mathbf{p}_1 \rangle \langle \mathbf{p}_3 | \mathbf{p}_2 \rangle \\
&\quad + \langle 0 | a(\mathbf{p}_3) a^\dagger(\mathbf{p}_1) \{ [a(\mathbf{p}_4), a^\dagger(\mathbf{p}_2)] + a^\dagger(\mathbf{p}_2) a(\mathbf{p}_4) \} | 0 \rangle \\
&= \langle \mathbf{p}_4 | \mathbf{p}_1 \rangle \langle \mathbf{p}_3 | \mathbf{p}_2 \rangle + (2\pi)^3 2p_2^0 \delta(\mathbf{p}_4 - \mathbf{p}_2) \langle 0 | a(\mathbf{p}_3) a^\dagger(\mathbf{p}_1) | 0 \rangle \\
&= \langle \mathbf{p}_4 | \mathbf{p}_1 \rangle \langle \mathbf{p}_3 | \mathbf{p}_1 \rangle + \langle \mathbf{p}_4 | \mathbf{p}_2 \rangle \langle \mathbf{p}_3 | \mathbf{p}_1 \rangle.
\end{aligned}$$

5(a).

(b)

$$\begin{aligned}
\Delta_F(x) &= \frac{1}{(2\pi)^4} \int_{\mathcal{C}_F} \frac{d^4 k}{k^2 - m^2} e^{-ik \cdot x} \\
&= \frac{1}{(2\pi)^4} \int_{\mathcal{C}_F} \frac{d^4 k}{(k^0)^2 - \mathbf{k}^2 - m^2} e^{-ik \cdot x} \\
&= \frac{1}{(2\pi)^4} \int_{\mathcal{C}_F} \frac{d^4 k}{(k^0 - \sqrt{\mathbf{k}^2 + m^2})(k^0 + \sqrt{\mathbf{k}^2 + m^2})} e^{-ik \cdot x}.
\end{aligned}$$

For $x^0 > 0$ we may close the contour towards $k^0 \rightarrow -i\infty$.

$$\Delta_F(x) = - \frac{1}{(2\pi)^4} \int_{\mathcal{C}^+} \frac{d^4 k}{(k^0 - \sqrt{\mathbf{k}^2 + m^2})(k^0 + \sqrt{\mathbf{k}^2 + m^2})} e^{-ik \cdot x}.$$

The $-ve$ sign is because on closing the contour $\mathcal{C}_F$ the sense of rotation is clockwise, and $\mathcal{C}^+$ is defined to be traversed anticlockwise. By Cauchy's Theorem, the value of the k^0 integral is then $(2\pi i)$ times the residue of the pole at $k^0 = \sqrt{\mathbf{k}^2 + m^2}$, and we have

$$\begin{aligned}\Delta_F(x) &= -\frac{i}{(2\pi)^3} \int \frac{d^3\mathbf{k}}{2\sqrt{\mathbf{k}^2 + m^2}} e^{-i(\sqrt{\mathbf{k}^2 + m^2}x^0 - \mathbf{k}\cdot\mathbf{x})} \\ &= -\frac{i}{(2\pi)^3} \int \frac{d^3\mathbf{k}}{2k^0} e^{-ik\cdot x},\end{aligned}$$

where in the last equation $k^0 = \sqrt{\mathbf{k}^2 + m^2}$.

For $x^0 < 0$ we may close the contour towards $k^0 \rightarrow +i\infty$.

$$\Delta_F(x) = \frac{1}{(2\pi)^4} \int_{\mathcal{C}^-} \frac{d^4k}{(k^0 - \sqrt{\mathbf{k}^2 + m^2})(k^0 + \sqrt{\mathbf{k}^2 + m^2})} e^{-ik\cdot x}.$$

The value of the k^0 integral is then $(2\pi i)$ times the residue of the pole at $k^0 = -\sqrt{\mathbf{k}^2 + m^2}$, and we have

$$\begin{aligned}\Delta_F(x) &= -\frac{i}{(2\pi)^3} \int \frac{d^3\mathbf{k}}{2\sqrt{\mathbf{k}^2 + m^2}} e^{-i(-\sqrt{\mathbf{k}^2 + m^2}x^0 - \mathbf{k}\cdot\mathbf{x})} \\ &= -\frac{i}{(2\pi)^3} \int \frac{d^3\mathbf{k}}{2\sqrt{\mathbf{k}^2 + m^2}} e^{i(\sqrt{\mathbf{k}^2 + m^2}x^0 - \mathbf{k}\cdot\mathbf{x})} \\ &= -\frac{i}{(2\pi)^3} \int \frac{d^3\mathbf{k}}{2k^0} e^{ik\cdot x},\end{aligned}$$

where we have used the fact that the integral over $\mathbf{k}$ is unchanged by the change $\mathbf{k} \rightarrow -\mathbf{k}$, and in the last equation $k^0 = \sqrt{\mathbf{k}^2 + m^2}$. Writing

$$\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x)$$

where

$$\begin{aligned}\phi^{(+)}(x) &= \frac{1}{(2\pi)^3} \int \frac{d^3\mathbf{p}}{2p^0} a(\mathbf{p}) e^{-ip\cdot x} \\ \phi^{(-)}(x) &= \frac{1}{(2\pi)^3} \int \frac{d^3\mathbf{p}}{2p^0} a^\dagger(\mathbf{p}) e^{ip\cdot x},\end{aligned}$$

we have

$$\begin{aligned}[\phi^{(+)}(x), \phi^{(-)}(0)] &= \frac{1}{(2\pi)^6} \int \frac{d^3\mathbf{p}}{2p^0} \frac{d^3\mathbf{p}'}{2p'^0} [a(\mathbf{p}), a^\dagger(\mathbf{p}')] e^{-ip\cdot x} \\ &= \frac{1}{(2\pi)^6} \int \frac{d^3\mathbf{p}}{2p^0} \frac{d^3\mathbf{p}'}{2p'^0} (2\pi)^3 2p^0 \delta(\mathbf{p} - \mathbf{p}') e^{-ip\cdot x} \\ &= \frac{1}{(2\pi)^3} \int \frac{d^3\mathbf{p}}{2p^0} e^{-ip\cdot x} \\ &= \frac{1}{(2\pi)^3} \int \frac{d^3\mathbf{k}}{2k^0} e^{-ik\cdot x},\end{aligned}$$

and similarly

$$[\phi^{(+)}(0), \phi^{(-)}(x)] = \frac{1}{(2\pi)^3} \int \frac{d^3\mathbf{k}}{2k^0} e^{ik \cdot x}.$$

So for $x^0 > 0$

$$\Delta_F(x) = -i[\phi^{(+)}(x), \phi^{(-)}(0)],$$

while for $x^0 < 0$

$$\Delta_F(x) = -i[\phi^{(+)}(0), \phi^{(-)}(x)].$$

(c)

$$\begin{aligned} \langle 0 | \phi(x) \phi(0) | 0 \rangle &= \langle 0 | \phi^{(+)}(x) \phi^{(-)}(0) | 0 \rangle \\ &= \langle 0 | [\phi^{(+)}(x), \phi^{(-)}(0)] | 0 \rangle, \end{aligned}$$

since $\phi^{(+)}|0\rangle = 0$, and similarly

$$\langle 0 | \phi(0) \phi(x) | 0 \rangle = \langle 0 | [\phi^{(+)}(0), \phi^{(-)}(x)] | 0 \rangle.$$

So

$$\begin{aligned} i\Delta_F(x) &= \langle 0 | \phi(x) \phi(0) | 0 \rangle \quad \text{if } x^0 > 0. \\ i\Delta_F(x) &= \langle 0 | \phi(0) \phi(x) | 0 \rangle \quad \text{if } x^0 < 0, \end{aligned}$$

i.e.

$$\langle 0 | T\{\phi(x), \phi(0)\} | 0 \rangle = i\Delta_F(x).$$