MATH431 - Modern Particle Physics
Set Work: Sheet 4;

1. Consider the Lagrangian

$$L(q(t), \dot{q}(t), t)$$

and variation of

$$q(t) \quad \rightarrow \quad q(t) \;+\; \delta(q(t)) = q(t) \;+\; \epsilon h(q(t), t)$$

where ϵ is an infinitesimal constant. Show that if the Lagrangian is invariant under the variation then the charge defined as

$$\epsilon Q \equiv \frac{\partial L}{\partial \dot{q}} \delta q$$

is conserved.

2. The potential function of a two-dimensional harmonic oscillator is

$$V(x, y) \;=\; \frac{1}{2}\, k\, (x^2 + y^2)^2\, .$$

(i) Write down the Lagrangian of this system.

(ii) Write down the Euler-Lagrange equations of motion.

(iii) Write down the Hamiltonian.

(iv) Write down the Lagrangian and Hamiltonian in polar coordinates (r, ϕ) with $(x = r\cos\phi\,,\; y = r\sin\phi)\,.$

(v) How many constants of the motion are there? What are they?

3. Prove

$$\frac{\partial \rho}{\partial t} + \text{div}\mathbf{j} = 0,$$

$$\text{where} \quad \mathbf{j} = -\,\frac{i\hbar}{2m}\,[\psi^* \nabla \psi - \psi \nabla \psi^*]$$

using the standard non-relativistic Schrödinger equation.

4. Assume a five-dimensional space-time $(t, \mathbf{x}, y)$, where $x = (t, \mathbf{x})$ are the usual four-dimensional space-time coordinates and y is the coordinate of an additional compact extra dimension, $-R/2 \leq y \leq R/2$.

Consider the free Klein-Gordon equation (KG) in this space-time,

$$(\partial_\sigma \partial^\sigma + m^2)\phi \;=\; 0\,,$$

where $\sigma = 0, 1, 2, 3, 4$ and $\mathbf{x} = \{x^1, x^2, x^3\}$ and $y = \{x^4\}$, *i.e.* $\partial_\sigma \partial^\sigma \equiv \partial_\mu \partial^\mu - \partial^2/\partial y^2$, with $\partial_\mu \partial^\mu = \partial_0^2 - \nabla^2$ the usual d'Alembert operator. The general solution of KG equation is obtained by a Fourier expansion and by imposing appropriate boundary conditions.

Show that

$$\phi(x, y) \;=\; \sum_{n=1}^{\infty} \phi_n(x)\, \mathrm{cs}\left(\frac{n\pi y}{R}\right) ,$$

where $\mathrm{cs}(n\pi y/R) = \cos(n\pi y/R)$ if n is odd and $\mathrm{cs}(n\pi y/R) = \sin(n\pi y/R)$ for even n, is a solution of the KG equation, provided that the Fourier coefficients $\phi_n(x)$ are solutions of a four-dimensional Klein-Gordon equation. Determine the mass spectrum as seen by a four-dimensional observer, and show that for $m = 0$ the masses are equally spaced. What is this infinite set of massive particles called?